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ERGODICITY, CONTINUITY, AND ANALYTICITY OF COUNTABLE MARKOV CHAINS

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ABSTRACT. General criteria are given for the ergodicity, recurrence, and transience of countable Markov chains. Conditions are given for the continuity and analyticity, in the parameter, of the stationary probabilities of families of such chains. A complete classification is obtained for a certain class of random walks in $\mathbf{Z}_+^n$ for $n = 2, 3$, and sufficient conditions are given for ergodicity and transience for arbitrary n. All criteria are closely connected with the well-known criterion of Foster for the ergodicity of Markov chains. For the consideration of concrete examples, a wide generalization of this criterion is given in terms of semimartingales.

Bibliography: 26 titles.

CONTENTS

INTRODUCTION

In the present paper we give general ergodicity, recurrence, and transience criteria for countable Markov chains, and also conditions for continuity and analyticity, in the parameter, of stationary probabilities of families of such chains. These criteria are closely connected with Foster's well-known criteria [23] for the ergodicity of countable Markov chains. For probability theory they are developed in the most natural way in the language of semimartingales or, which is equivalent, in the language of "test" functions. These functions are also called Ljapunov functions. In [3] by means of these functions ergodicity and stability problems of diffusion processes are studied.

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These general criteria turned out to be necessary for the classification of random walks in $\mathbf{Z}_+^n = \{(z_1, \ldots, z_n)\colon z_i \geqslant 0, \text{ integral}\}$ of a definite form (maximally homogeneous and with restriction on the jumps (cf. below)). The solution of the latter problem is our main purpose. Besides, on it we can test the effectiveness of the general criteria for Markov chains and martingale sequences developed in the paper.

The significance of the problem is explained, in particular, by the following circumstances:

1. The difficulties in obtaining existence and uniqueness conditions for solutions of partial differential or convolution equations in domains with edges are well known. In the language of probability theory (for stochastic operators) this problem reduces to the classification of the corresponding Markov chain. Here the corresponding operator is not a Fredholm operator; however, the probabilistic character of the problem enables us to penetrate deeper than in the case of general functional equations. For a detailed exposition of the results concerning general functional equations in $\mathbf{Z}_+^n$, see [7].

2. Many nonstandard Markov queueing problems can be presented in the form of random walks in some piecewise linear domain on the discrete lattice $\mathbf{Z}_+^n$ with various conditions on the faces. (A queueing problem is said to be Markov if all parameters are exponentially distributed.) Examples for such problems can be found in [4], [14] and [15].

3. A connection with the quantum n-body problem is obvious, where all difficulties are basically surmounted for small n, which also holds in our problem (for more detail, see [7]).

However, the general criteria obtained in the paper go beyond the scope of the problem of a random walk. For instance, we obtain necessary and sufficient conditions for the continuity of stationary probabilities and also sufficient conditions for the analyticity of a family of Markov chains.

Let us now go into the content of the work. In §1 of Chapter I we consider a sequence of real random variables $S_0, S_1, \ldots,$ concerning which we assume that $S_0 = \text{const}$, and there exists a constant d such that for all $n = 1, 2, \ldots$ we have $|S_n - S_{n-1}| \leqslant d$ with probability 1. From the sequence $\{S_n\}$ a subsequence $\{S_{N_i}\}$ is selected, where N_i is a random index sequence such that $|N_i - N_{i-1}| < r$ with probability 1 for some $r > 0$ and any i. We claim that, according as the sequence $\{S_{N_i}\}$ is a strict supermartingale or submartingale, the mean first passage time of a fixed boundary by the sequence $\{S_n\}$ is either finite or infinite.

The fundamental theorem of §2 is Theorem 1.4, which is a generalization of Foster's famous theorem concerning the necessary and sufficient ergodicity condition for countable Markov chains [23]. It is formulated in terms of the existence of a so-called "test" function.

In Chapter II we consider a homogeneous irreducible aperiodic Markov chain L with discrete time and state set $\mathbf{Z}_+^n = \{(z_1, \ldots, z_n)\colon z_i \geqslant 0, \text{ integral}\}$ concerning which the homogeneity conditions and the boundedness condition of jumps are assumed to be satisfied (cf. the definitions in §1 of Chapter II). We assume that can classify the random walks in $\mathbf{Z}_+^m$, where $m \leqslant n - 1$, in the sense of ergodicity and transience, and for ergodic walks we can calculate stationary probabilities. Then in R_+^n we give an algorithm for the construction of a vector field V. In terms of the existence of a "test" function $f(\alpha)$, $\alpha \in R_+^n$, satisfying certain conditions concerning V, sufficient conditions are given for the ergodicity and transience of L.

In §2 of Chapter II an explicit construction method for the "test" function $f(\alpha)$ is given, which enables us to classify random walks in $\mathbf{Z}_+^n$ for $n \leqslant 3$. By means of V we construct a deterministic process $\Psi(t)$ and prove that ergodicity of L and finiteness of the first passage time of the origin by the process $\Psi(t)$ are equivalent.

In Chapter III we consider a family of homogeneous irreducible aperiodic Markov chains $\{L^\nu\}$ with discrete time and countable state set $A = \{0, 1, \dots\}$; $\nu \in D$, where D is an open subset of the real line. Under the assumption that the transition probabilities are continuous in the parameter, we give a necessary and sufficient condition for the continuity of stationary probabilities. This condition is close to the compactness of the distribution family. In §2 of Chapter III sufficient conditions are given for the continuity of stationary probabilities of the family $\{L^\nu\}$ in terms of "test" functions.

In §1 of Chapter IV we consider the Markov chain family $\{L^\nu\}$ on the set $A = \{0, 1, \dots\}$ for $\nu \in D$. We denote by $\mathfrak{X}(A, \Sigma)$ the Banach space of real countably additive measures on (A, Σ) with norm equal to the total variation (Σ is the σ-algebra of all subsets of A). It is easy to see that $\mathfrak{X}(A, \Sigma) \equiv l_1(A)$. Let $B(\mathfrak{X})$ be the Banach algebra of bounded linear operators in $\mathfrak{X}(A, \Sigma)$. To L^ν there corresponds an operator $P_\nu \in B(\mathfrak{X})$ with norm 1.

In Theorem 4.1 sufficient conditions are given for the existence and analyticity of $r(\nu) = \lim_{n\to\infty} P^n(\nu)x$, where x belongs to some set $M \subset \mathfrak{X}(A, \Sigma)$ and $P(\nu)$ depends on ν analytically.

In §2 of Chapter IV we give sufficient conditions for the analyticity of the stationary probabilities of a family $\{L^\nu\}$ of countable chains in terms of "test" functions. The proof of this theorem is based on Theorem 4.1.

In §3 of Chapter III and §3 of Chapter IV the results of §§1 and 2 of those chapters are used for the study of continuity and analyticity of stationary probabilities of the random walk family $\{L^\nu\}$ in $\mathbf{Z}_+^n$.

CHAPTER I. ERGODICITY CRITERIA FOR COUNTABLE MARKOV CHAINS

§1. A lemma for semimartingale sequences

Let us consider a sequence $S_0, S_1, \dots$ of real random variables, concerning which we shall assume that $S_0 = \text{const}$ and there exists a constant d such that for all $n = 1, 2, \dots$ we have

$$|S_n - S_{n-1}| \leqslant d \tag{1.1}$$

with probability 1.

Denote by t the first passage time of the boundary b, i.e. $t = 0$ if $S_0 \leqslant b$, and $t = n$ if $S_n \leqslant b$ but $S_i > b$ for $i = 0, 1, \dots, n - 1$.

LEMMA 1.1. *If for some $\varepsilon > 0$ the inequality**

$$M(S_n/S_{n-1}, \ldots, S_0) \leqslant S_{n-1} - \varepsilon \tag{1.2}$$

holds for all n with probability 1, then for any $\delta_1 < \varepsilon$ there exist constants c and $\delta_2 > 0$ such that for any n

$$P(S_n > -\delta_1 n) < c\cdot\exp\{-\delta_2 n\},$$
$$M(t) < \infty. \tag{1.3}$$

In other words, the mean first passage time of b for a uniformly strict semimartingale is finite.

PROOF. Set $y_n = S_n - S_{n-1}$. It follows from the generalized Tchebycheff inequality that

$$P(S_n > 0) = P\left(\sum_{k=1}^{n} y_k - S_0\right) \leqslant \exp\{-hS_0\}\cdot M\left[\exp\left\{h\sum_{k=1}^{n} y_k\right\}\right]. \tag{1.4}$$

As follows from condition (1.1), inequality (1.4) is true for any $h > 0$. Let us estimate $M[\exp\{h\Sigma_1^n y_k\}]$. Let h be subject to the condition $0 < h < 1/d$. Then we have

$$\exp\{hy_k\} < 1 + hy_k + \frac{3}{2}(hy_k)^2,$$
$$M\{\exp\{hy_k\}/S_0, S_1, \ldots, S_n\} < M\left\{1 + hy_k + \frac{3}{2}(hy_k)^2/S_0, S_1, \ldots, S_n\right\}$$
$$\leqslant 1 - h\varepsilon + \frac{3}{2}h^2d^2$$

with probability 1. Therefore, choosing h sufficiently small, for some $\delta > 0$ and any k we have

$$M\{\exp\{h\cdot y_k\}/S_0, S_1, \ldots, S_{k-1}\} < \exp\{-\delta\} \tag{1.5}$$

with probability 1. It follows from (1.5) that

$$M\left[\exp\left\{h\sum_{k=1}^{n} y_k\right\}\right] = M\left[\prod_{k=1}^{n}\exp\{hy_k\}\right]$$
$$= M\left[\prod_{k=1}^{n-1}\exp\{hy_k\}\cdot M\{\exp\{hy_k\}/S_0, \ldots, S_{n-1}\}\right]$$
$$< M\left[\prod_{k=1}^{n-1}\exp\{hy_k\}\cdot\exp\{-\delta\}\right].$$

* *Editor's note.* In the Russian literature, MX denotes the expectation of X, and $M(A/B)$ denotes the conditional expectation of A, given B.

Analogous calculations show that

$$M\left[\exp\left\{h\sum_{k=1}^{n} y_k\right\}\right] < \exp\{-n\delta\}.$$

Setting $c = \exp\{-hS_0\}$, we obtain

$$P(S_n > 0) < c\cdot\exp\{-n\delta\}. \tag{1.6}$$

Take an arbitrary $\delta_1 < \varepsilon$. We introduce a random sequence $\{\tilde{S}_n\}$ by putting $\tilde{S}_n = S_n + n\delta_1$. Then

$$\begin{aligned}M(\tilde{S}_n/\tilde{S}_{n-1}, \ldots, \tilde{S}_0) - \tilde{S}_{n-1} &= M(S_n/S_{n-1}, \ldots, S_0)\\ &+ n\delta_1 - S_{n-1} - (n-1)\delta_1 \leqslant -\varepsilon + \delta_1 = -\varepsilon_1 < 0.\end{aligned} \tag{1.7}$$

It follows from (1.7) that

$$P(\tilde{S}_n > 0) < c_1\cdot\exp\{-n\delta_2\} \tag{1.8}$$

for some c_1, $\delta_2 > 0$ and all n. Therefore,

$$P(S_n > -\delta_1 n) = P(\tilde{S}_n > 0) < c_1\cdot\exp\{-n\delta_2\}. \tag{1.9}$$

Consequently, the first assertion of the lemma is proved. Moreover,

$$M(t) = \sum_{n=1}^{\infty} nP(S_0 > b,\ S_1 > b,\ \ldots, S_{n-1} > b,\ S_n \leqslant b) \leqslant \sum_{n=1}^{\infty} nP(S_{n-1} > b). \tag{1.10}$$

The exponential estimate (1.3) implies the convergence of the series (1.10). The lemma is proved.

In the course of the proof we obtained exponential estimates, necessary for the study of the analyticity of the Markov chain families in Chapter IV. Similar estimates are obtained for semimartingale sequences in [25] and [26].

The following lemma sharpens the assertion of Lemma 1.1 in a certain sense.

Let $\{N_i\}$ be a random sequence of positive integers ($i = 1, 2, \ldots$) such that for some $r > 0$ and any i we have

$$1 \leqslant N_i - N_{i-1} < r \tag{1.11}$$

with probability 1.

LEMMA 1.2. *Assume that for some* $\varepsilon > 0$ *and all* i *the inequality*

$$M(S_{N_i}/S_{N_{i-1}}, S_{N_{i-2}}, \ldots, S_0) \leqslant S_{N_{i-1}} - \varepsilon \tag{1.12}$$

holds with probability 1. *Then for any* $\delta_1 < \varepsilon$ *there exist constants* c *and* $\delta > 0$ *such that for any* n

$$\begin{gathered}P(S_n > -\delta_1 n) < c\cdot\exp\{-\delta n\},\\ M(t) < \infty.\end{gathered} \tag{1.13}$$

In other words, the mean first passage time to b *for the random sequence from which a strict semimartingale subsequence is selected is finite.*

Proof. Form a random sequence $\{\omega_i\}$ by setting $\omega_i = S_{N_i}$ and $\omega_0 = S_0$. The sequence $\{\omega_i\}$ satisfies the hypotheses of Lemma 1.1. Therefore, for any $\delta_1 < \varepsilon$ there exist constants c_1, $\delta_2 > 0$ such that for any i

$$P(\omega_i > -\delta_1 i) < c_1 \cdot \exp\{-\delta_2 i\}. \tag{1.14}$$

It follows easily from (1.14) that there exist constants c_2, $\delta_3 > 0$ such that for any i

$$P(\omega_i > -\delta_1 i - dr) < c_2 \cdot \exp\{-\delta_3 i\}. \tag{1.15}$$

Consider the event $A_n = (S_n > -\delta_1 n)$. It follows from (1.1) and (1.11) that

$$A_n \subset \bigcup_{m=\left[\frac{n}{r}\right]}^{n} (\omega_m > -\delta_1 m - dr).$$

Consequently, taking account of (1.15), we obtain

$$P(A_n) = P(S_n > -\delta_1 n) \leqslant \sum_{m=\left[\frac{n}{r}\right]}^{n} P(\omega_m > -\delta_1 m - dr)$$

$$\leqslant c_2 \sum_{m=\left[\frac{n}{r}\right]}^{n} \exp\{-\delta_3 n\}.$$

It follows immediately from the latter inequality that there exist constants c, $\delta > 0$ such that for any i

$$P(S_n > -\delta_1 n) < c \cdot \exp\{-\delta n\}. \tag{1.16}$$

As in Lemma 1.1, (1.16) implies the finiteness of the mean first passage time of the sequence S_n to b. The lemma is proved.

Let $\{N_i\}$ be the random index sequence introduced above.

Lemma 1.3. *Assume that for all i and some $\varepsilon > 0$ the inequality*

$$M\,(S_{N_i}/S_{N_{i-1}}, \ldots, S_0) > S_{N_{i-1}} + \varepsilon \tag{1.17}$$

holds with probability 1. *If $S_0 > b + dr$, then $P(t = \infty) > 0$.*

Proof. It is sufficient to prove that for some m there exists $\sigma > 0$ such that

$$P(S_m > b,\ S_{m+1} > b,\ \ldots, S_n > b) > \sigma. \tag{1.18}$$

for $S_0 > b + dr$. Obviously we have

$$P(S_m > b,\ S_{m+1} > b,\ \ldots) > 1 - P\left(\bigcup_{n=m}^{\infty} (S_n \leqslant b)\right) > 1 - \sum_{n=m}^{\infty} P(S_n \leqslant b). \tag{1.19}$$

Analogously as in Lemma 1.2, but reversing the inequalities, we can prove the existence of constants $\delta > 0$ and m such that for any n

$$P(S_n \leqslant b) < c\exp\{-\delta n\}. \tag{1.20}$$

The convergence of $\sum_{n=1}^{\infty} e^{-\delta n}$ implies the existence of $\sigma > 0$ and m such that

$$\sum_{n=m}^{\infty} P(S_n \leqslant b) < 1 - \sigma. \tag{1.21}$$

Inequalities (1.19) and (1.21) imply (1.18). The lemma is proved.

For completeness we formulate the following lemma whose proof we omit.

LEMMA 1.4. *If* $M(S_n/S_{n-1}, \ldots, S_0) = S_{n-1}$, *for all n, then* $M(t) = \infty$.

In what follows we shall prove ergodicity criteria for countable Markov chains and random walks. The results of this section enable us to remove the requirement that the random walks be Markov.

§2. Criteria for countable Markov chains

Consider a homogeneous Markov chain L with discrete time, countable state set $B = \{0, 1, \ldots\}$ and transition probabilities p_{ij}, $i, j \in B$. We assume that L has a single essential class of states and it is aperiodic. By p_{ij}^n we shall denote the transition probability from i to j in n steps ($p_{ij}^1 = p_{ij}$).

On the set of nonnegative integers let an integer-valued positive function $k(i) = k_i$ be given. On the state set B define the Markov chain $\tilde{L}$ by the transition probabilities $\tilde{p}_{ij} = p_{ij}^{k_i}$ (i.e. the transition probability of $\tilde{L}$ from the state i to the state j is equal to the transition probability of L from i to j in k_i steps).

THEOREM 1.1. *If there exists at least one recurrent state in* $\tilde{L}$, *then L is recurrent.*

PROOF. Let the state j of $\tilde{L}$ be recurrent. Then $\sum_{r=1}^{\infty} \tilde{p}_{jj}^r = \infty$. We write the terms of this sum in detail:

$$\tilde{p}_{jj}^r = \sum_{i_1, i_2, \ldots, i_{r-1}} \tilde{p}_{ji_1} \tilde{p}_{i_1 i_2} \ldots \tilde{p}_{i_{r-1} j},$$

$$\tilde{p}_{i_l i_{l+1}} = \sum_{t_1, \ldots, t_{k(i_l)-1}} p_{i_l t_1} p_{t_1 t_2} \ldots p_{t_{k(i_l)-1}, i_{l+1}}.$$

Therefore $\sum_{r=1}^{\infty} \tilde{p}_{jj}^r$ will consist of different products $p_{ji}, p_{i_1 i_2} \ldots p_{i_m j}$, which differ from each other in either the order of factors or the factors themselves. The sum $\sum_{n=1}^{\infty} p_{jj}^n$ can be represented in an analogous way, and it contains all the terms of $\sum_{r=1}^{\infty} \tilde{p}_{jj}^r$. Consequently $\sum_{n=1}^{\infty} p_{jj}^n = \infty$, and L is recurrent. The theorem is proved.

The following theorem is well known.

THEOREM 1.2 (see [8]). *In order that the irreducible Markov chain L be recurrent it is sufficient that there exist a sequence* $\{y_i\}$ *such that*

$$\sum_{j=0}^{\infty} p_{ij} y_j \leqslant y_i, \quad i \notin A, \quad y_i \to \infty \quad as \quad i \to \infty, \tag{1.22}$$

and the set A be finite.

The following theorem generalizes Theorem 1.2.

THEOREM 1.3. *In order that the irreducible Markov chain L be recurrent it is sufficient that there exist a positive integer-valued sequence* $\{k_i\}$ *and a sequence* $\{y_i\}$

such that

$$\sum_{j=0}^{\infty} p_{ij}^{k_i} y_j \leqslant y_i, \quad i \notin A, \quad y_i \to \infty \quad as \quad i \to \infty. \tag{1.23}$$

The set A is finite and the chain $\tilde{L}$ formed in the way described above is irreducible.

PROOF. Let the hypotheses of the theorem be satisfied. Then

$$\sum_{j=1}^{\infty} \tilde{p}_{ij} y_j \leqslant y_i, \quad i \notin A, \quad y_i \to \infty \quad \text{as} \quad i \to \infty.$$

By Theorem 1.2 this reduces to the recurrence of $\tilde{L}$, and, moreover, by Theorem 1.1 to the recurrence of L.

THEOREM 1.4. *The irreducible aperiodic Markov chain L is ergodic if and only if there exist a positive sequence $\{y_i\}$ and an integer-valued positive sequence $\{k_i\}$ such that for some $\varepsilon > 0$ and a finite set A the following system of inequalities holds*:

$$\sum_{j=0}^{\infty} p_{ij}^{k_i} y_j \leqslant y_i - \varepsilon k_i, \quad i \notin A,$$
$$\sum_{j=0}^{\infty} p_{ij}^{k_i} y_j < \infty, \quad i \in A. \tag{1.24}$$

PROOF. We prove the sufficiency. As in Theorem 1.1, we shall consider the Markov chain $\tilde{L}$ with transition probabilities $\tilde{p}_{ij} = p_{ij}^{k_i}$. The inequalities (1.24) can be rewritten in the following way:

$$\sum_{j=0}^{\infty} \tilde{p}_{ij} y_j \leqslant y_i - \varepsilon k_i, \quad i \notin A,$$
$$\sum_{j=0}^{\infty} \tilde{p}_{ij} y_j < \infty, \quad i \in A. \tag{1.25}$$

Set $\max_{i\in A} \sum_{j=0}^{\infty} \tilde{p}_{ij} y_j = \lambda$. By recurrence we define

$$y_i^{n+1} = \sum_{j=0}^{\infty} \tilde{p}_{ij} y_j^n; \quad y_i^1 = y_i. \tag{1.26}$$

It follows from this definition that $y_i^n > 0$ for any i and n. We have

$$y_i^2 \leqslant y_i - k_i\varepsilon, \quad i \notin A;$$
$$y_i^2 \leqslant \lambda, \quad i \in A;$$

$$y_i^3 = \sum_{j=0}^{\infty} \tilde{p}_{ij} y_j^2 = \sum_{j\in A} \tilde{p}_{ij} y_j^2 + \sum_{j\notin A} \tilde{p}_{ij} y_j^2 \leqslant \lambda \sum_{j\in A} \tilde{p}_{ij} + \sum_{j\notin A} \tilde{p}_{ij} (y_j - k_j\varepsilon)$$
$$= \lambda \sum_{j\in A} \tilde{p}_{ij} + y_i^2 - \sum_{j\in A} \tilde{p}_{ij} y_j - \varepsilon \sum_{j=0}^{\infty} \tilde{p}_{ij} k_j + \varepsilon \sum_{j\in A} \tilde{p}_{ij} k_j$$
$$\leqslant y_i^2 + \tilde{p}_{iA} (\lambda + \varepsilon\lambda_1) - \varepsilon \sum_{j=0}^{\infty} \tilde{p}_{ij} k_j,$$

where

$$\max_{j\in A} k_j = \lambda_1; \quad \sum_{j\in A} \tilde{p}_{ij} = p_{iA}; \quad \sum_{j\in A} \tilde{p}_{ij}^n = \tilde{p}_{iA}^n.$$

We prove the following inequality by induction:

$$y_i^n \leq y_i^{n-1} + \tilde{p}_{iA}^{n-2}(\lambda + \varepsilon\lambda_1) - \varepsilon \sum_{j=0}^{\infty} \tilde{p}_{ij}^{n-2} k_j. \tag{1.27}$$

We assume that this inequality is true for $n = m$, and prove it for $n = m + 1$. We have

$$y_i^m \leq y_i^{m-1} + \tilde{p}_{iA}^{m-2}(\lambda + \varepsilon\lambda_1) - \varepsilon \sum_{j=0}^{\infty} \tilde{p}_{ij}^{m-2} k_j,$$

$$y_i^{m+1} = \sum_{j=0}^{\infty} \tilde{p}_{ij} y_j^m \leq \sum_{j=0}^{\infty} \tilde{p}_{ij} \left[y_j^{m-1} + \tilde{p}_{jA}^{m-2}(\lambda + \varepsilon\lambda_1) - \varepsilon \sum_{l=0}^{\infty} \tilde{p}_{jl}^{m-2} k_l \right]$$

$$= y_i^m + \tilde{p}_{iA}^{m-1}(\lambda + \varepsilon\lambda_1) - \varepsilon \sum_{j=0}^{\infty} \tilde{p}_{ij}^{m-1} k_j,$$

as asserted.

From (1.27) we obtain

$$y_i^{n+2} \leq y_i^2 + (\lambda + \varepsilon\lambda_1) \sum_{r=1}^{n} \tilde{p}_{iA}^r - \varepsilon \sum_{r=1}^{n} \sum_{j=0}^{\infty} \tilde{p}_{ij}^r k_j. \tag{1.28}$$

Let $\xi_0, \xi_1, \ldots$ and $\tilde{\xi}_0, \tilde{\xi}_1, \ldots$ be sequences of random variables corresponding to the chains L and $\tilde{L}$, with $\xi_0 = \tilde{\xi}_0 = i$. Then

$$M(k(\tilde{\xi}_r)) = \sum_{j=0}^{\infty} \tilde{p}_{ij}^r k_j.$$

Inequality (1.28) implies that

$$\sum_{r=1}^{n} \tilde{p}_{iA}^r \geq -\frac{y_i^2}{\lambda + \varepsilon\lambda_1} + \frac{\varepsilon}{\lambda + \varepsilon\lambda_1} \sum_{r=1}^{n} M(k(\tilde{\xi}_r)).$$

Take an arbitrary $c > 0$. Then

$$\tilde{p}_{iA}^r = P(\tilde{\xi}_r \in A) = P\left\{(\tilde{\xi}_r \in A) \cap \left(\sum_{l=1}^{r} k(\tilde{\xi}_l) < cr\right)\right\} + P\left\{(\tilde{\xi}_r \in A) \cap \left(\sum_{l=1}^{r} k(\tilde{\xi}_l) \geq cr\right)\right\}.$$

Write

$$B_n^c = \sum_{r=1}^{n} P\left\{(\tilde{\xi}_r \in A) \cap \left(\sum_{l=1}^{r} k(\tilde{\xi}_l) < cr\right)\right\};$$

$$B_n^c \geq -\frac{y_i^2}{\lambda + \varepsilon\lambda_1} + \frac{\varepsilon}{\lambda + \varepsilon\lambda_1} \sum_{r=1}^{n} M(k(\tilde{\xi}_r)) - \sum_{r=1}^{n} P\left\{(\tilde{\xi}_r \in A) \cap \left(\sum_{l=1}^{r} k(\tilde{\xi}_l) > cr\right)\right\}.$$

$$B_n^c \geq -\frac{y_i^2}{\lambda + \varepsilon\lambda_1} + \frac{\varepsilon}{\lambda + \varepsilon\lambda_1} \sum_{r=1}^{n} M(k(\tilde{\xi}_r)) - \sum_{r=1}^{n} P\left(\sum_{l=1}^{r} k(\tilde{\xi}_l) > cr\right). \tag{1.29}$$

There are two possibilities:

$$1.\ \varlimsup_{n\to\infty} \frac{\sum_{r=1}^{\infty} M(k(\tilde{\xi}_r))}{n} > \frac{\lambda + \varepsilon\lambda_1}{\varepsilon} + \varepsilon_1$$

for some $\varepsilon_1 > 0$, and

$$2.\ \varlimsup_{n\to\infty} \frac{\sum_{r=1}^{n} M(k(\tilde{\xi}_r))}{n} \leqslant \frac{\lambda + \varepsilon\lambda_1}{\varepsilon}.$$

Consider the first case:

$$\varlimsup_{n\to\infty} \frac{B_n^c}{n} \geqslant \varlimsup_{n\to\infty} \frac{-\dfrac{y_i^2}{\lambda+\varepsilon\lambda_1} + \dfrac{\varepsilon}{\lambda+\varepsilon\lambda_1}\sum_{r=1}^{n} M(k(\tilde{\xi}_r)) - n}{n} \geqslant \frac{\varepsilon_1\varepsilon}{\lambda+\varepsilon\lambda_1} > 0.$$

Consider the second case. Then there is an r_0 such that for any $r > r_0$

$$\frac{\sum_{l=1}^{r} M(k(\tilde{\xi}_l))}{r} < \frac{2(\lambda+\varepsilon\lambda_1)}{\varepsilon} = \varepsilon_2. \tag{1.30}$$

By the Tchebycheff inequality we have

$$P\left(\sum_{l=1}^{r} k(\tilde{\xi}_l) > cr\right) \leqslant \frac{M\left(\sum_{l=1}^{r} k(\tilde{\xi}_l)\right)}{cr} = \frac{\sum_{l=1}^{r} M(k(\tilde{\xi}_l))}{cr}. \tag{1.31}$$

Using (1.30) and (1.31), we obtain

$$\frac{B_n^c}{n} \geqslant \frac{y_i^2}{\lambda+\varepsilon\lambda_1}\frac{1}{n} + \frac{\varepsilon}{(\lambda+\varepsilon\lambda_1)n}\sum_{r=1}^{n} M(k(\tilde{\xi}_r)) - \frac{1}{n}\sum_{r=1}^{n}\frac{\sum_{l=1}^{r} M(k(\tilde{\xi}_l))}{cr}$$

$$\geqslant -\frac{y_i^2}{\lambda+\varepsilon\lambda_1}\frac{1}{n} + \frac{\varepsilon}{\lambda+\varepsilon\lambda_1} - \frac{1}{n}\sum_{r=1}^{r_0}\frac{\sum_{l=1}^{r} M(k(\tilde{\xi}_l))}{cr}$$

$$-\frac{1}{n}\sum_{r=r_0+1}^{n}\frac{\sum_{l=1}^{r} M(k(\tilde{\xi}_l))}{cr} \geqslant \frac{y_i^2}{\lambda+\varepsilon\lambda_1}\frac{1}{n} - \frac{1}{n}\sum_{r=1}^{r_0}\frac{\sum_{l=1}^{r} M(k(\tilde{\xi}_l))}{cr_0}$$

$$+\frac{\varepsilon}{\lambda+\varepsilon\lambda_1} - \frac{1}{n}\frac{(n-r_0)\varepsilon_2}{c}.$$

Consequently,

$$\varliminf_{n\to\infty} \frac{B_n^c}{n} \geqslant \frac{\varepsilon}{\lambda+\varepsilon\lambda_1} - \frac{\varepsilon_2}{c}.$$

Since as c we can take any positive number, by choosing it sufficiently large, we obtain

$$\underline{\lim}_{n\to\infty} \frac{B_n^c}{n} > \delta_1 > 0$$

for some $\delta_1 > 0$.

In both cases we have obtained that

$$\overline{\lim}_{n\to\infty} \frac{B_n^c}{n} > \delta > 0$$

for some $\delta > 0$. The quantity

$$B_n^c = \sum_{r=1}^{n} P\left\{(\tilde{\xi}_r \in A) \cap \left(\sum_{l=1}^{r} k(\tilde{\xi}_l) < cr\right)\right\}$$

can be represented in the form of a sum of products $p_{ii_1}p_{i_1i_2}\cdots p_{i_lj}$ $(j \in A)$, as has already been done for the expansion $\sum_{r=1}^{\infty} \tilde{p}_{jj}^r$ in the proof of Theorem 1.1. Moreover, the number of factors in each term of this sum does not exceed $[cn] + 1$, and these terms differ from each other by either the order of the factors or the factors themselves. The sum $\sum_{r=1}^{[cn]+1} p_{iA}^r$ can be represented in an analogous way, and will contain all terms of the expanion of B_n^c. Therefore, for any n we have

$$\sum_{r=1}^{[cn]+1} p_{iA}^r > B_n^c. \tag{1.32}$$

It follows from (1.32) that

$$\overline{\lim}_{n\to\infty} \frac{\sum_{r=1}^{[nc]+1} p_{iA}^r}{[nc]+1} > \frac{1}{c+1}\overline{\lim}_{n\to\infty} \frac{B_n^c}{n} > \frac{\delta}{c+1} > 0. \tag{1.33}$$

(1.33) implies the ergodicity of the irreducible Markov chain. By the same token, the sufficiency of the hypotheses of the theorem is proved.

Now we assume that the chain L is ergodic. Set $k_i \equiv 1$, $i = 0, 1, \ldots$. Then the fundamental theorem of [23] implies the existence of a sequence $\{y_i\}$ such that all the hypotheses of the theorem are satisfied. The theorem is proved.

THEOREM 1.5. *In order that the irreducible aperiodic Markov chain L be ergodic it is sufficient that there exist an integer-valued sequence $\{k_i\}$ such that*

$$\sup_{i=0,1,2,\ldots} k_i = k < \infty, \qquad \inf_{i=0,1,2,\ldots} k_i \geqslant 1,$$

and a positive sequence $\{y_i\}$ such that for some $\varepsilon > 0$ and some finite set A

$$\sum_{j=0}^{\infty} p_{ij}^{k_i} y_j \leqslant y_i - \varepsilon, \quad i \notin A;$$

$$\sum_{j=0}^{\infty} p_{ij}^{k_i} y_j < \infty, \quad i \in A. \tag{1.34}$$

The proof follows from Theorem 1.4 in an obvious way.

Assume that on the set of nonnegative integers a real function $y = \{y_i\}$ $(i \in B)$ is given in such a way that $y_i > 0$ for any i, and for some $d > 0$ the inequality $|y_i - y_j| > d$ implies that $p_{ij} = 0$, where the p_{ij} are the transition probabilities of L. On the same set B let a bounded integer-valued positive function k_i $(i \in B)$ be given, where $\sup_{i \in B} k_i = k < \infty$. Under these hypotheses we have the following theorem.

THEOREM 1.6. *Assume the positive sequence $\{y_i\}$ is such that*

$$\sum_{j=0}^{\infty} p_{ij}^{k_i} y_j > y_i + \varepsilon \tag{1.35}$$

for some ε, $c > 0$ and for all i belonging to the nonempty set $A_c = \{i: y_i > c\}$. Then the Markov chain L is transient.

PROOF. It follows from (1.35) that for any $N > 0$ there exists an n such that $y_n > N$. Consider the sequence $\xi_0, \xi_1, \ldots$ of random variables constituting the chain L. Let $\xi_0 = \alpha_0 > c + dk$. From the sequence $\{\xi_i\}$ let us form the sequence $\{S_i\}$ by putting $S_n = y(\xi_n) = y_{\xi_n}$. Denote by t the random time of entrance of the sequence S_n to c. From the sequence $\{\xi_i\}$ we also form a random integer-valued sequence $\{N_i\}$ by putting

$$N_0 = k(\xi_0) = k_{\alpha_0}; \quad N_i = N_{i-1} + k(\xi_i).$$

Then (1.35) implies that

$$M\,(S_{N_i}/S_{N_{i-1}} > c) \geqslant S_{N_{i-1}} + \varepsilon \tag{1.36}$$

with probability 1. Applying Lemma 1.3, from (1.36) we obtain that $t = \infty$ with positive probability. This means that the first passage time of the process $\{\xi_n\}$ to the set $B \setminus A_c$ is infinite with positive probability. Consequently, the chain L is transient. The theorem is proved.

THEOREM 1.7. *Assume that the positive sequence $\{y_i\}$ is such that*

$$\sum_{j=0}^{\infty} p_{ij} y_j \geqslant y_i \tag{1.37}$$

for some $c > 0$ and all i belonging to the nonempty set $A_c = \{i: y_i > c\}$. Then L is not ergodic.

The proof is easy if we use Lemma 1.4.

CHAPTER II. ERGODICITY OF RANDOM WALKS

§1. Sufficient conditions for ergodicity and transience of random walks in $\mathbf{Z}_+^n$

Consider a homogeneous irreducible aperiodic Markov chain L with discrete time whose set of states is the set $\mathbf{Z}_+^n = \{(z_1, \ldots, z_n): z_i \geqslant 0, \text{ integral}\}$. Let $p_{\alpha\beta}^k$ $(\alpha, \beta \in \mathbf{Z}_+^n)$ be the k-step transition probabilities of L, and $\mathbf{M}^k(\alpha) = (M_1^k(\alpha), \ldots, M_n^k(\alpha))$ the vector of the mean jump from the point α in k steps; $p_{\alpha\beta}^1 = p_{\alpha\beta}$; $\mathbf{M}^1(\alpha) = \mathbf{M}(\alpha)$.

By $\Lambda = \{i_1, \ldots, i_k\}$ we denote a k-tuple of natural numbers from 1 to n $(1 \leq k \leq n, i_1 < \cdots < i_k)$. Let $R^n_+ = \{(r_1, \ldots, r_n): r_i \geq 0, \text{ real}\}$. By B^Λ we denote an arbitrary face in R^n_+, i.e.

$$\Lambda \subseteq \{1, 2, \ldots, n\},\ B^\Lambda = \{(r_1, \ldots, r_n): r_i > 0,\ i \in \Lambda;\ r_i = 0,\ i \notin \Lambda\}.$$

for some $\Lambda \subseteq \{1, \ldots, n\}$. By $|\Lambda|$ we understand the dimension of Λ. Set

$$B^\Lambda_{ct} = \{(r_1, \ldots, r_n): r_i > c,\ i \in \Lambda;\ \ r_i \leqslant t,\ i \notin \Lambda\}.$$

In what follows, we shall consider only bounded maximally homogeneous random walks in $\mathbf{Z}^n_+$, i.e., random walks satisfying the following conditions:

HOMOGENEITY CONDITION: *There exists $c > 0$ such that for any Λ, for any vector $a = (a_1, \ldots, a_n)$ such that $a_i \geq 0$, $1 \leq i \leq n$, and $a_j = 0$ for $j \notin \Lambda$, and for all $\alpha \in B^\Lambda_{cc} \cap \mathbf{Z}^n_+$ we have*

$$p_{\alpha\beta} = p_{\alpha+a,\beta+a} \quad (\beta \in \mathbf{Z}^n_+).$$

BOUNDEDNESS OF JUMPS. *For any α the number of β such that $p_{\alpha\beta} \neq 0$ is finite.*

By the homogeneity condition, this is equivalent to the following: *there exists $d > 0$ such that $p_{\alpha\beta} = 0$ for $\|\alpha - \beta\| > d$.*

For any face B^Λ $(\Lambda \neq \{1, \ldots, n\})$ we choose an arbitrary point $a \in \mathbf{Z}^n_+ \cap B^\Lambda \cap B^\Lambda_{cc}$. Through this point we draw a plane $C^\Lambda \subset \mathbf{Z}^n_+$ of dimension $n - |\Lambda|$, perpendicular to B^Λ. We consider the Markov chain L^Λ with set C^Λ of states (which we shall call *induced by L*) and one-step transition probabilities

$${}_\Lambda p_{\alpha\beta} = p_{\alpha\beta} + \sum_{\beta' \neq \beta} p_{\alpha\beta'},\quad \alpha, \beta \in C^\Lambda,$$

where summation is performed for all $\beta' \in \mathbf{Z}^n_+$, such that the straight line connecting β' and β is perpendicular to C^Λ. It follows from the homogeneity condition that the definition of L^Λ does not depend on the choice of the point $a \in \mathbf{Z}^n_+ \cap B^\Lambda \cap B^\Lambda_{cc}$.

In the sequel we consider only chains satisfying the following condition A:

CONDITION A1. *For any Λ the chain L^Λ is irreducible and aperiodic.*

If the chain L^Λ is ergodic, let $\Pi^\Lambda(\gamma)$ $(\gamma \in C^\Lambda)$ be its stationary probabilities. We introduce the vector $v^\Lambda = (v^\Lambda_1, \ldots, v^\Lambda_n)$, by setting

$$v^\Lambda_i = 0 \quad \text{for} \quad i \notin \Lambda;$$

$$v^\Lambda_i = \sum_{\gamma \in C^n} \pi^\Lambda(\gamma) M_i(\gamma) \quad \text{for} \quad i \in \Lambda.$$

For $\Lambda = \{1, 2, \ldots, n\}$ we set

$$v^\Lambda = \mathbf{M}(\alpha), \text{ where } \in B^{\{1,2,\ldots,n\}}_{cc} \cap \mathbf{Z}^n_+.$$

In this way we have a finite collection of vectors V^Λ. (From the boundedness of jumps it follows that $\max\|v^\Lambda\| < \infty$.) For nonergodic chains L^Λ the vectors V^Λ are not defined.

CONDITION A2. *We have* $\|v^\Lambda\| \neq 0$ *for all* Λ *for which the vectors* v^Λ *are introduced.*

Now we formulate and prove an assertion relating to countable Markov chains. Let us consider an arbitrary homogeneous irreducible aperiodic Markov chain Q with discrete time and countable state set $B = \{b\}$. Besides, for every point $b \in B$ let a probability distribution F_b be given on the set of real vectors of dimension k such that the random variables corresponding to F_b are uniformly bounded for all $b \in B$. Let M_b denote the mean value of F_b. Let a sequence of random vectors $(\varphi_n, \xi_n) \equiv \varphi_n(\xi_n)$ be given such that $\xi_1, \xi_2, \ldots$ is the initial chain Q and for any given $\xi_i = b_i$, $i = 1, 2, \ldots,$ the random variables φ_i, $i = 1, 2, \ldots,$ are independent and have distribution functions F_{b_i}.

LEMMA 2.1. *Assume that the chain Q is ergodic and $\xi_1 = a$ at time $t = 1$. Then for any $\varepsilon > 0$*

$$P\left(\left\| \sum_{i=1}^{n} \varphi_i(\xi_i) - n\mathbf{v}\right\| > n\varepsilon\right) \to 0 \quad as\ n \to \infty, \tag{2.1}$$

where $\mathbf{v} = \sum_{b\in B} \mathbf{M}_b \pi_b$ *(the π_b are stationary probabilities of Q).*

PROOF. We prove that

$$M(\varphi_n(\xi_n)) \to \mathbf{v} \quad \text{as} \quad n \to \infty. \tag{2.2}$$

Indeed,

$$M(\varphi_n(\xi_n)) = \sum_{b\in B} M\{\varphi_n(\xi_n)/\xi_n = b\} \cdot P(\xi_n = b).$$

$$M\{\varphi_n(\xi_n)/\xi_n = b\} = \mathbf{M}_b. \tag{2.3}$$

It follows from the ergodicity of Q that $P(\xi_n = b) \to \Pi_b$ as $n \to \infty$. Taking into account that $\sup_{b\in B}\|\mathbf{M}_b\| < \infty$, we conclude that $M(\varphi_n(\xi_n)) \to \mathbf{v}$ as $n \to \infty$. Therefore, instead of (2.1), it is sufficient to show that for any $\varepsilon > 0$ we have

$$P\left(\left\|\sum_{i=1}^{n} \varphi_i(\xi_i) - \sum_{i=1}^{n} M(\varphi_i(\xi_i))\right\| > n\varepsilon\right) \to 0 \quad \text{as} \quad n \to \infty. \tag{2.4}$$

Apply the Tchebycheff inequality to the sum of the ith components of the vectors $\varphi_n(\xi_m)$:

$$P\left(\left|\sum_{m=1}^{n} \varphi_m^l(\xi_m) - \sum_{i=1}^{n} M\left((\varphi_m^l(\xi_i)\right)\right| > n\varepsilon\right) \leqslant \frac{D\left[\sum_{m=1}^{n} (\varphi_m^l(\xi_m) - M(\varphi_m^l(\xi_m))\right]}{n^2\varepsilon^2}. \tag{2.5}$$

It follows from the uniform boundedness of the variables $\varphi_m^l(\xi_m)$ that for the proof of (2.4) it is sufficient to show that

$$M\left[(\varphi_m^l(\xi_m) - M(\varphi_m^l(\xi_m)))(\varphi_n^l(\xi_n) - M(\varphi_n^l(\xi_n)))\right] \to 0 \tag{2.6}$$

as $|i-j| \to \infty$ for any $1 \leqslant l \leqslant k$. This will hold in the case where

$$M(\varphi_m^l(\xi_m)\varphi_n^l(\xi_n)) \to M(\varphi_m^l(\xi_m))M(\varphi_n^l(\xi_n))$$

as $|n-m| \to \infty$. The latter assertion follows from the independence of the random variables $\varphi_m(\xi_m)$ and $\varphi_n(\xi_n)$ for fixed ξ_m and ξ_n and the ergodicity of Q. The lemma is proved.

By means of Lemma 2.1 we can prove the following assertion concerning random walks. Let ξ^m be the location of the random walk in the mth step, and let $\xi^0 = \alpha$.

LEMMA 2.2. *Assume that either L^Λ is ergodic for $|\Lambda| < n$ or $\Lambda = \{1, 2, \ldots, n\}$. Then for any R_2, ε, and $\sigma > 0$ there are a natural number m and an $R_1 > 0$ such that for any point $\alpha \in B^\Lambda_{R_1R_2} \cap \mathbf{Z}^n_+$*

$$P\{\|\xi^m - (\alpha + mv^\Lambda)\| > m\varepsilon\} < \sigma; \quad \xi_0 = \alpha. \tag{2.7}$$

PROOF. In order to apply Lemma 2.1, consider the chain Q^Λ whose states are all the one-step transitions of L^Λ, i.e. the pairs (a_i, a_j) $(a_i, a_j \in C^\Lambda)$. The transition probabilities of Q^Λ are defined as

$$P_{(a_1a_2)(a_3a_4)} = \begin{cases} 0, & a_2 \neq a_3, \\ P_{a_2a_4}, & a_2 = a_3. \end{cases}$$

Let Π_a $(a \in C^\Lambda)$ be stationary probabilities of L^Λ, and $\Pi_{(a,b)}$ $((a,b) \in C^\Lambda \times C^\Lambda)$ stationary probabilities of Q^Λ. It is obvious that

$$\pi_{(a,b)} = \pi_a {}_\Lambda p_{ab}. \tag{2.8}$$

Let $\xi_0 = \alpha, \xi_1, \xi_2, \ldots$ be the sequence of random variables corresponding to the random walk L, and let $\Lambda = (i_1, \ldots, i_k)$. We introduce a sequence of random variables φ_m by setting

$$\varphi_m = (\xi^{m+1}_{i_1} - \xi^m_{i_1}, \xi^{m+1}_{i_2} - \xi^m_{i_2}, \ldots, \xi^{m+1}_{i_k} - \xi^m_{i_k}).$$

This random sequence satisfies the hypotheses of Lemma 2.1. Using it, we obtain the assertion of Lemma 2.2.

On the set $\mathbf{Z}^n_+$ let a real function $f(\alpha)$ $(\alpha \in \mathbf{Z}^n_+)$ be given such that the condition $|f_\alpha - f_\beta| > d$ implies that $p_{\alpha\beta} = 0$ for some $d > 0$. Write

$$B^\Lambda_R = \{(x_1, \ldots, x_n) : x_i > R,\ i \in \Lambda\}.$$

LEMMA 2.3. *Assume that the chain L^Λ is not ergodic, and there exist a set $B^\Lambda_{R_1R_2}$ and a function $m(\alpha)$ defined on the set $B^\Lambda_{R_1R_2}$ and a function $m(\alpha)$ defined on the set $(B^\Lambda_{R_1} \setminus B^\Lambda_{R_1R_2}) \cap \mathbf{Z}^n_+$ and taking values in the set of natural numbers such that for all $\alpha \in (B^\Lambda_{R_1} \setminus B^\Lambda_{R_1R_2}) \cap \mathbf{Z}^n_+$ the inequality*

$$\sum_{\beta \in Z^n_+} p^{m(\alpha)}_{\alpha\beta} f_\beta - f_\alpha < -\varepsilon \tag{2.9}$$

holds for some $\varepsilon > 0$, *and*

$$\sup_{\alpha \in (B^{\Lambda}_{R_1} \setminus B^{\Lambda}_{R_1 R_2}) \cap Z^n_+} m(\alpha) = m < \infty. \tag{2.10}$$

Then there exist a set B^{Λ}_R *and a function* $n(\alpha)$ ($\alpha \in B^{\Lambda}_R$), *taking values in the set of natural numbers such that for all* $\alpha \in B^{\Lambda}_R \cap \mathbf{Z}^n_+$ *the inequality*

$$\sum_{\beta \in Z^n_+} p^{n(\alpha)}_{\alpha\beta} f_\beta - f_\alpha < -\varepsilon_1 \tag{2.11}$$

holds for some $\varepsilon_1 > 0$, *and*

$$\sup_{\alpha \in B^{\Lambda}_R \cap Z^n_+} n(\alpha) = n < \infty; \quad n(\alpha) \equiv m(\alpha), \ \alpha \in B^{\Lambda}_R \cap B^{\Lambda}_{R_1 R_2}.$$

PROOF. Let $\xi_0 = \alpha, \xi_1, \xi_2, \ldots$ be the sequence of random variables corresponding to the chain L. Form the random index sequence N_i by setting $N_0 = m(\alpha_0)$ and $N_i = N_{i-1} + m(\xi_{i-1})$ (for $\alpha \in \mathbf{Z}^n_+ \setminus B^{\Lambda}_{R_1 R_2}$, we complete the definition of $m(\alpha)$ by setting $m(\alpha) = 1$). The sequence ξ_{N_i} forms a Markov chain $\tilde{L}$. (§2 of Chapter I deals with such chains in more detail.) $\tilde{L}^{\Lambda}$ is the Markov chain induced by $\tilde{L}$ on the state set C^{Λ}. It is obvious that if $\{\xi_i^{\Lambda}\}$ is the sequence of random variables corresponding to L^{Λ}, then the sequence $\{\xi^{\Lambda}_{N_i}\}$ corresponds to $\tilde{L}^{\Lambda}$. The nonergodicity of L^{Λ} implies that of $\tilde{L}^{\Lambda}$. Therefore, for any $\sigma > 0$ there exist $R \gg R_1$ and $t > 0$ such that for all r with $(R - R_1)/m > r > t$, we have

$$P(\tilde{\xi}_r \in B^{\Lambda}_{R_1 R_2}) > 1 - \sigma, \tag{2.12}$$

provided that $\xi_0 = \alpha \in B_{RR_2}$. Taking account of (2.9), from (2.12) we obtain that

$$M\{f(\tilde{\xi}_r) - f(\tilde{\xi}_{r-1})\} = M\{f(\tilde{\xi}_r) - f(\tilde{\xi}_{r-1})/\tilde{\xi}_{r-1} \notin B^{\Lambda}_{R_1 R_2}\} \cdot P(\tilde{\xi}_{r-1} \bar{\in} B^{\Lambda}_{R_1 R_2})$$
$$+ M\{f(\tilde{\xi}_r) - f(\tilde{\xi}_{r-1})/\tilde{\xi}_{r-1} \in B^{\Lambda}_{R_1 R_2}\} \cdot P(\tilde{\xi}_{r-1} \in B^{\Lambda}_{R_1 R_2}) \leqslant dm\sigma - \varepsilon(1 - \sigma).$$

Consequently, if σ is chosen sufficiently small (then R has to be chosen sufficiently large), we obtain

$$M\{f(\tilde{\xi}_r) - f(\tilde{\xi}_{r-1})\} < -\sigma_1$$

for some $\sigma_1 > 0$ and $(R - R_1)/m > r > t$; $\xi_0 = \alpha \in B^{\Lambda}_{RR_2}$. Moreover,

$$M\{f(\tilde{\xi}_r)\} - f_\alpha = \sum_{i=1}^{r} M\{f(\tilde{\xi}_i) - f(\tilde{\xi}_{i-1})\} < tdm - \sigma_1(r - t).$$

Therefore, for sufficiently large r we have

$$M\{f(\tilde{\xi}_r)\} - f(\alpha) < -md - \varepsilon_1 \tag{2.13}$$

for some $\varepsilon_1 > 0$. The inequality

$$M\{f(\xi_k)\} < \sup_{k/m < r < k} M\{f(\tilde{\xi}_r)\} + md \tag{2.14}$$

is obvious. Therefore, for sufficiently large k

$$M\{f(\xi_k)\} - f_\alpha \leqslant -\varepsilon_1. \tag{2.15}$$

Inequality (2.15) is equivalent to (2.11) if we put $n(\alpha) = k$ for $\alpha \in B^{\Lambda}_{RR_2} \cap \mathbf{Z}^n_+$. For $\alpha \in (B^{\Lambda}_R \setminus B^{\Lambda}_{RR_2}) \cap \mathbf{Z}^n_+$ we set $n(\alpha) \equiv m(\alpha)$. The lemma is proved.

We have previously introduced a finite collection $\{\mathbf{v}^{\Lambda}\}$ of vectors. To each point α belonging to a face B^{Λ} such that L^{Λ} is ergodic (such a face B^{Λ} will be said to be ergodic) we assign the vector $\mathbf{v}(\alpha) = \mathbf{v}^{\Lambda}$. For the points $\alpha \in B^{\{1,2,\dots,n\}}$ we set $\mathbf{v}(\alpha) = \mathbf{v}^{\{1,2,\dots,n\}}$. In this way we obtain a vector field V, which may not be defined on certain faces.

CONDITION B. *For some $\delta, b, p > 0$ there exists a function $f(\alpha)$ ($\alpha \in R^n_+$) having the following properties*:

1. $f(\alpha) \geqslant 0, \quad \alpha \in R^n_+$.
2. $f(\alpha) - f(\beta) \leqslant b\|\alpha - \beta\|, \quad \alpha, \beta \in R^n_+$.

3. *For any Λ such that L^{Λ} is ergodic and for $\Lambda = \{1, \dots, n\}$ and all $\alpha \in B^{\Lambda} \cap B^{\Lambda}_{pp}$*

$$f(\alpha + \vec{v}(\alpha)) - f(\alpha) < -\delta.$$

CONDITION B′. *For some $\delta, b, t, p > 0$ there exist a function $f(\alpha)$ ($\alpha \in R^n_+$) and a nonempty set $T \subset R^n_+$ satisfying the following conditions*:

1. $f(\alpha) \geqslant 0, \quad \alpha \in R^n_+$.
2. $f(\alpha) - f(\beta) \leqslant b\|\alpha - \beta\|, \quad \alpha, \beta \in R^n_+$.
3. $f(\alpha) \geqslant t, \quad \alpha \in T$;

$f(\alpha) < t, \quad \alpha \in R^n_+ \setminus T$.

4. *For any Λ such that L^{Λ} is ergodic and for $\Lambda = \{1, \dots, n\}$ and all $\alpha \in B^{\Lambda} \cap B^{\Lambda}_{pp} \cap T$ we have*

$$f(\alpha + \mathbf{v}(\alpha)) - f(\alpha) > \delta.$$

THEOREM 2.1. *If the vector field V satisfies condition* B, *then the random walk L is ergodic. If condition* B′ *is satisfied, then L is transient.*

PROOF. Assume that a function $f(\alpha)$ ($\alpha \in R^n_+$) satisfying condition B exists. As follows from Theorem 1.5, for the ergodicity of L it is sufficient to show the existence of a function $m(\alpha)$ ($\alpha \in \mathbf{Z}^n_+$), taking values in the set of natural numbers, such that

$$\sup_{\alpha \in \mathbf{Z}^n_+} m(\alpha) = m < \infty$$

and for all $\alpha \in \mathbf{Z}^n_+$, except some finite set, the inequality

$$\sum_{\beta \in \mathbf{Z}^n_+} p^{m(\alpha)}_{\alpha\beta} f_\beta - f_\alpha < -\varepsilon_1 \tag{2.16}$$

is satisfied for some $\varepsilon_1 > 0$. Let $\Lambda = \{1, 2, \ldots, n\}$. It follows from Lemma 2.2 that for any $\varepsilon, \sigma > 0$ there exist m^Λ and R^Λ such that inequality (2.7) is satisfied for all $\alpha \in B^\Lambda_{R^\Lambda}$. Therefore, if we take ε and σ sufficiently small, and take account of the boundedness of the jumps of the random walk and the properties of the functions f (condition B), we obtain that for any $\alpha \in B^\Lambda_{R^\Lambda} \cap \mathbf{Z}^n_+$ inequality (2.16) is satisfied for some $\varepsilon_1 > 0$ if we set

$$m(\alpha) \equiv m^\Lambda \quad \text{for} \quad \alpha \in B^\Lambda_{R^\Lambda} \cap \mathbf{Z}^n_+.$$

We continue the construction of $m(\alpha)$ ($\alpha \in \mathbf{Z}^n_+$) by induction. Assume that for all Λ, $|\Lambda| = k < n$, there exist sets $B^\Lambda_{R^\Lambda}$ and a function $m(\alpha)$ with values in the set of natural numbers such that

$$\sup_{\alpha \in \left(\bigcup_{|\Lambda|=k} B^\Lambda_{R^\Lambda}\right) \cap \mathbf{Z}^n_+} m(\alpha) < \infty$$

and for all $\alpha \in (\bigcup_{|\Lambda|=k} B^\Lambda_{R^\Lambda}) \cap \mathbf{Z}^n_+$ inequality (2.16) is satisfied for some $\varepsilon_1 > 0$. Take Λ_1 such that $|\Lambda_1| = k - 1$. It follows from the definition of the sets $B^\Lambda_{R_1 R_2}$ and B^Λ_R that there exist $R_1^{\Lambda_1}, R_2^{\Lambda_2} > 0$, and together with them sets $B^{\Lambda_1}_{R_1^{\Lambda_1} R_2^{\Lambda_2}}$, such that

$$B^{\Lambda_1}_{R_1^{\Lambda_1}, R_2^{\Lambda_1}} \subset \bigcup_{|\Lambda|=k} B^\Lambda_{R^\Lambda}. \tag{2.17}$$

Let L_1^Λ be nonergodic. Then, applying Lemma 2.3 and using (2.17), we conclude that there exist R^{Λ_1} and a function $n(\alpha)$ with values in the set of natural numbers such that

$$\sup_{\alpha \in \mathbf{Z}^n_+} n(\alpha) < \infty,$$

$$n(\alpha) \equiv m(\alpha) \quad \text{for} \quad \alpha \in B^{\Lambda_1}_{R_1 R_2} \cap B^{\Lambda_1}_{R^{\Lambda_1}} \cap \mathbf{Z}^n_+$$

and

$$\sum_{\beta \in \mathbf{Z}^n_+} p^{n(\alpha)}_{\alpha\beta} f_\beta - f_\alpha < -\varepsilon^{\Lambda_1} \tag{2.18}$$

for all $\alpha \in B^{\Lambda_1}_{R^{\Lambda_1}}$ and some $\varepsilon_1^{\Lambda_1} > 0$.

Let L^Λ be ergodic. In this case we use Lemma 2.2, and obtain the same result as in the case where L^Λ is nonergodic. Sorting out all Λ with $|\Lambda| = k - 1$, we obtain that the conditions which we assumed to be satisfied for all Λ with $|\Lambda| = k$ will also be satisfied for all Λ with $|\Lambda| = k - 1$.

Hence, we have proved by induction that for all Λ such that $|\Lambda| = 1$ there exist sets $B^\Lambda_{R^\Lambda}$ for some $R^\Lambda > 0$ and a function $m(\alpha)$ with values in the set of natural numbers such that

$$\sup_{\alpha \in \bigcup_{|\Lambda|=1} B^\Lambda_{R^\Lambda} \cap \mathbf{Z}^n_+} m(\alpha) < \infty$$

and for all $\alpha \subset \bigcup_{|\Lambda|=1} B^\Lambda_{R^\Lambda}$ inequality (2.16) is satisfied for some $\varepsilon_1 > 0$. This completes the proof of the ergodicity of the random walk, if we take into account that $\mathbf{Z}^n_+ \setminus \bigcup_{|\Lambda|=1} B^\Lambda_{R^\Lambda}$ is a finite set.

Now assume that there exists a function $f(\alpha)$ ($\alpha \in R^n_+$), satisfying condition B′. It follows from Theorem 1.6 that for the proof of the transience of L it is sufficient to show the existence of a function $m(\alpha)$ ($\alpha \in \mathbf{Z}^n_+$), with values in the set of natural numbers, such that

$$\sup_{\alpha \in \mathbf{Z}^n_+} m(\alpha) = m < \infty$$

and for all $\alpha \in T$, except some finite set, the inequality

$$\sum_{\beta \in \mathbf{Z}^n_+} p^{m(\alpha)}_{\alpha\beta} f_\beta - f_\alpha > \varepsilon_1 \tag{2.19}$$

is satisfied for some $\varepsilon_1 > 0$. The proof of the existence of $m(\alpha)$ can be carried out by induction analogously and in the same succession as in the ergodic case. The theorem is proved.

§2. Classification of random walks in $\mathbf{Z}^2_+$ and $\mathbf{Z}^3_+$

In this section we give a classification of random walks in $\mathbf{Z}^n_+$ ($n \leq 3$) from the viewpoint of ergodicity and transience. As the results of the preceding section show, to prove the ergodicity or transience of a random walk L in $\mathbf{Z}^n_+$ it is necessary to classify all induced chains L^Λ of dimension less than n, and to calculate the vectors v^Λ introduced previously for the ergodic chains L^Λ. Then for the vector field V corresponding to $\{v^\Lambda\}$ we have to construct a function $f(\alpha)$ ($\alpha \in \mathbf{Z}^n_+$) satisfying either condition B or condition B′ of §1. A method is given for constructing this function, which also enables us to classify random walks in $\mathbf{Z}^n_+$ for $n \leq 3$.

The vector field V constructed above was defined only for points belonging to ergodic faces. We extend the definition of V to points belonging to nonergodic faces. To the point $x \in R^n_+$ we assign the vector $\mathbf{v}(x) = \mathbf{v}^\Lambda$ if and only if the following conditions are satisfied:

1. *The point x belongs to the closure of B^Λ.* (2.20a)

2. $x + \delta\mathbf{v}^\Lambda \in R^n_+$ *for all sufficiently small* $\delta > 0$. (2.20b).

We note that the field V may now be multi-valued. Since it is difficult to study the cases of zero recurrence of random walks by the method applied below, we shall assume the following condition to be satisfied.

CONDITION A3. $v^\Lambda_i \neq 0$ *for* $i \in \Lambda$.

As the following theorems show, in this case all chains will be either ergodic or transient for random walks in $\mathbf{Z}^n_+$, where $n \leq 3$.

Let $n = 1$.

THEOREM 2.2. *If* $v^{(1)} = M(\alpha) < 0$ ($\alpha \in B^{(1)}_c$), *then L is ergodic. If* $v^{(1)} > 0$, *then L is transient.*

This result is well known; therefore we omit its proof, although it can be carried out easily by the method of "test" functions.

Let $n = 2$.

LEMMA 2.4. *The vector field V constructed as above is nonempty and single valued at each point $x \in R_+^2 \setminus 0$. Besides, the same vector is assigned to any two points belonging to the same face B^Λ.*

PROOF. To each point $x \in B^{\{1,2\}}$ a unique vector $\mathbf{v}^{\{1,2\}}$ is assigned. Let x belong to $B^{\{i_1\}}$, i.e. to a one-dimensional face. There are two possibilities: either the chain $L^{\{i_1\}}$ is ergodic, or it is transient (the case of zero recurrence is excluded, as follows from condition A3 and Theorem 2.1). In the first case, the vector $\mathbf{v}^{\{i_1\}}$ is assigned to the point $x \in B^{\{i_1\}}$. The point x belongs to the closure of $B^{\{1,2\}}$. However, because of the ergodicity of $L^{\{i_1\}}$, by Theorem 2.1 we have $v_{i_2}^{\{1,2\}} < 0$, and therefore condition (20b) is not satisfied for the vector $\mathbf{v}^{\{1,2\}}$, and it is not assigned to the point x. If $L^{\{i_1\}}$ is transient, then by Theorem 2.1 we have $v_{i_2}^{\{1,2\}} > 0$, and consequently conditions (20a) and (20b) are satisfied; the (unique) vector $\mathbf{v}^{\{1,2\}}$ is assigned to the point x. The lemma is proved.

For the vector field V in R_+^2 we construct, in a natural way, a deterministic process $\psi(t)$ for which V is the field of velocities.

THEOREM 2.3. *If for any point $x \in R_+^2 \setminus 0$ the first passage time $\tau(x)$ of $\psi(t)$ to the origin is finite when $\psi(0) = x$, then the random walk L is ergodic. Otherwise L is transient.*

PROOF. Let $\tau(x)$ be finite for any point $x \in R_+^2 \setminus 0$. On R_+^2 introduce a function $f(x)$ by putting $f(x) = \tau(x)$. From the properties of V proved in Lemma 2.1 and from purely geometrical considerations it follows that $f(x)$ satisfies condition B of the preceding section. Consequently, L is ergodic.

On the other hand, if $\tau(x)$ is infinite for at least one point x, then we can construct a function $f(x)$ on R_+^2 satisfying condition B′, which leads to the transience of L. We shall not discuss the construction of this function, since it has a purely geometric character.

By sorting out the possible cases, the finiteness condition for the first passage time of the origin by the process $\psi(t)$ can be rewritten in terms of the field V in the form of the following theorem.

THEOREM 2.3′. *The process $\psi(t)$ reaches the origin in a finite time for any initial state $\psi(0) = x \in R_+^2$ if and only if the following two conditions are satisfied:*

1. *There exists an i_1 such that $v_{i_1}^{\{1,2\}} < 0$.*
2. *$v_{i_2}^{\{i_1\}} < 0$ for every i_1 such that $v_{i_1}^{\{1,2\}} < 0$.*

We remark that in [4] and [12] a classification is given for random walks in $\mathbf{Z}_+^2$ with more stringent homogeneity conditions, where an analogous theorem is formulated in a more explicit form. However, if these homogeneity conditions are imposed on L, then V can be calculated in terms of mean jumps, and the form of Theorem 2.3′ becomes simpler.

Let us pass to the study of random walks in $\mathbf{Z}_+^3$.

LEMMA 2.5. *For every point $x \in R_+^3 \setminus 0$ the vector field V is nonempty. Moreover, single valuedness can be violated only on one of the three one-dimensional faces of the type $B^{\{i_1\}}$. In this case two vectors are assigned to the points of $B^{\{i_1\}}$. Besides, the same vectors are assigned to any two points belonging to the same face B^Λ.*

The proof is based on Theorem 2.3′ and can be carried out analogously to the proof of Lemma 2.4 based on Theorem 2.2.

If the field V is single-valued, then it gives rise, in a natural way, to a deterministic process $\psi(t)$ in $R^3_+ \setminus 0$ for which V is the velocity field. On the other hand, if the single-valuedness of V is violated on a one-dimensional face $B^{\{i_1\}}$, then we choose one of the vectors assigned to $B^{\{i_1\}}$. In this way we obtain two single-valued fields V_1 and V_2. For each field V_i we construct a deterministic process $\psi_i(t)$.

THEOREM 2.4. *If for any point $x \in R^3_+ \setminus 0$ the first passage time $\tau(x)$ of the origin by the process $\psi_i(t)$ for $\psi_i(0) = x$ is finite for at least one i, $i = 1, 2$, then L is ergodic. On the other hand, if at least one process $\psi_i(t)$ is unbounded for some initial state $\psi(0) = x$, then L is transient.*

PROOF. Let $\tau_i(x)$ be finite for any point $x \in R^3_+$. As in Theorem 2.3, on R^3_+ we introduce a function $f(x)$ by setting $f(x) = \tau_i(x)$. It is easy to prove that if V is single-valued, then this function satisfies condition B of the preceding section, and so L is ergodic. If the single-valuedness of V is violated on a one-dimensional face $B^{\{i_1\}}$, then the continuity of $f(x)$ is violated on the plane going through the face $B^{\{i_1\}}$ and the vector $\mathrm{v}^{\{1,2,3\}}$. However, this discontinuity is easily removable. For this, the values of $f(x)$ have to be multiplied by the corresponding factor on one side of the indicated plane. The function thus corrected satisfies condition B, which leads to the ergodicity of L in this case.

We shall not give a proof for the transience of L in the case where $\psi_i(t)$ is unbounded, since it can be carried out analogously.

By sorting out all possible cases, the boundedness condition for the reaching time of the origin for $\psi_i(t)$ can be rewritten in the form of the following theorem.

THEOREM 2.4′. *The process $\psi_i(t)$ reaches the origin in finite time for any initial state $\psi_i(0) = x \in R^3_+$ if and only if the following three conditions are satisfied*:

1. *There exists an i_1 such that $v_{i_1}^{\{1,2,3\}} < 0$.*

2. *For every i_1 such that $v_{i_1}^{\{1,2,3\}} < 0$, there exists an i_2 for which $v_{i_2}^{\{i_2,i_3\}} < 0$.*

3. *Either* a) *there exists an i_1 such that $L^{\{i_1\}}$ is ergodic, and for any i_1 such that $L^{\{i_1\}}$ is ergodic we have $v_{i_1}^{\{i_1\}} < 0$; or* b) *the chains $L^{\{1\}}$, $L^{\{2\}}$, $L^{\{3\}}$ are transient. If $v_2^{\{2,3\}} > 0$, then*

$$\left| \frac{v_2^{\{2,3\}}}{v_3^{\{2,3\}}} \frac{v_3^{\{1,3\}}}{v_1^{\{1,3\}}} \frac{v_1^{\{1,2\}}}{v_2^{\{1,2\}}} \right| < 1,$$

and if $v_2^{\{2,3\}} < 0$, then

$$\left| \frac{v_2^{\{2,3\}}}{v_3^{\{2,3\}}} \frac{v_3^{\{1,3\}}}{v_1^{\{1,3\}}} \frac{v_1^{\{1,2\}}}{v_2^{\{1,2\}}} \right| > 1$$

(*the case where the expression between the absolute value signs is equal to* 1 *is not considered*).

The proof of this theorem is of geometric character and uses properties of the vector field V. We shall not give it here. We note only the case where conditions 1, 2, and 3b are satisfied. In this case, the process $\psi_i(t)$ ($\psi_i(0) = x$) reaches one of the two-dimensional faces in a finite time and starts successively intersecting all two-dimensional and one-dimensional faces, without going to $B^{\{1,2,3\}}$. Condition 3b shows that $\psi_i(t)$ will twist to one side or the other and reach the origin in a finite amount of time.

We make a few remarks concerning the ergodicity of random walks in $\mathbf{Z}_+^n$ for $n > 4$. Theorem 2.1 gives only sufficient conditions for the ergodicity of random walks. As a result of this we can see that for ergodic walks in $\mathbf{Z}_+^n$, $n \geq 4$, the function $f(\alpha)$, $\alpha \in \mathbf{Z}_+^n$, satisfying condition B of §1 does not always exist. Apparently, we can also study random walks for higher dimensions by similar methods. In that case, instead of the deterministic process $\psi(t)$ constructed in §2 and ergodically equivalent to a random walk, a random process $\tilde{\psi}(t)$, also ergodically equivalent to a random walk, is constructed (for points belonging to ergodic faces it behaves like a deterministic process). With an increase of the dimension of the random walk, the construction of $\tilde{\psi}(t)$ becomes more complicated. In the present paper we have not given the construction of such a process for $n > 4$. The solution of this problem presents much difficulty, and one has to use the boundary theory of Markov processes in the discrete case developed, for example, in [13].

We remark that random walks on a discrete lattice in a half-strip or in $\mathbf{Z}_+^n \times B$, where B is a finite set, can be studied by our method if certain homogeneity conditions analogous to those introduced in §1 are satisfied. By the same token, under the assumption of homogeneity, the ergodicity criteria developed in [21] for Markov chains in the special phase space E_S can be essentially simplified.

Chapter III. Continuity of stationary probabilities for a family of Markov chains

§1. Formulation of the problem. Necessary and sufficient conditions for the continuity of stationary probabilities

Recently, problems about continuity of a family of Markov processes have been studied mainly in connection with queueing systems, for example, in [16] and [17]; the interpretation of the notion of continuity varies. In [17] sufficient conditions are given for the uniform closeness of the components of a Markov chain by the method of "test" functions. In the present chapter we study the continuity of stationary distributions for families of homogeneous irreducible and aperiodic Markov chains. In §1 we give a necessary and sufficient condition for the continuity of stationary probabilities, and in §2 constructive sufficient conditions for the continuity of stationary probabilities in terms of "test" functions.

Let us consider a family of homogeneous irreducible aperiodic Markov chains $\{L^\nu\}$ with discrete time and countable set of states $A = \{0, 1, \dots\}$ for $\nu \in D$, where D is an open subset of the real line. By $p_{ij}(t, \nu)$ we denote the t-step transition probability from the point i to the point j in L^ν. Everywhere in this chapter we assume that the $p_{ij}(1, \nu)$ are continuous in ν for all $\nu \in p$ and $i, j \in A$.

LEMMA 3.1. *The $p_{ij}(t, \nu)$ are continuous functions of ν $(\nu \in D)$ for every natural number t and all $i,j \in A$.*

PROOF. We prove the lemma by induction. For $n = 1$ the function $p_{ij}(1, \nu)$ is continuous in ν for any $i, j \in A$. We have

$$p_{ij}(n+1, \nu) = \sum_{k=0}^{\infty} p_{ik}(1, \nu)\, p_{kj}(n, \nu) = \sum_{k=0}^{\infty} \varphi_k(\nu). \tag{3.1}$$

It follows from the inclusion hypothesis that the $\varphi_k(\nu)$ are continuous functions of ν for any k. We have

$$\sum_{k=0}^{\infty} p_{ik}(\nu) \equiv 1 \quad (\nu \in D). \tag{3.2}$$

Consequently, the series (3.2) consisting of continuous functions converges uniformly with respect to $\nu \in D$, and $p_{kj}(n, \nu) < 1$ for any k, j, n, and $\nu \in D$. These two conditions lead to the uniform convergence of the series (3.1). The sum of a uniformly convergent series of continuous functions is a continuous function. Consequently, the $p_{ij}(n + 1, \nu)$ are continuous in ν. The lemma is proved.

On the set A let a family $\{\pi_j(\nu)\}, j \in A, \nu \in D$, of distributions be given, where D is some open subset of the real line. We have

$$\sum_{j\in A} \pi_j(\nu) = 1 \quad (\nu \in D).$$

DEFINITION. The family of distributions $\{\pi_j(\nu)\}$ $(j \in A, \nu \in D)$ *satisfies condition* (λ) *at the point* $\nu_0 \in D$ if for any $\varepsilon > 0$ there exist $\delta > 0$ and a finite set $B^\varepsilon \subset A$ such that

$$\sum_{j \in A\setminus B^\varepsilon} \pi_j(\nu) < \varepsilon \tag{3.3}$$

for all ν with $|\nu - \nu_0| < \delta$.

Let the chains L^ν be ergodic for every ν belonging to some neighborhood $U_0 \subset D$ of zero.

THEOREM 3.1. *The stationary probabilities $\pi_j(\nu)$ depend on ν continuously at $\nu = 0$ for all $j \in A$ if and only if the family of distributions $\{\pi_j(\nu)\}$ satisfies condition (λ) at the point $\nu = 0$.*

Before proving this we make the following remark. Following Ju. V. Prohorov [22], we form the metric space $D(A)$. For this we define the distance $L(\mu_1, \mu_2)$ between any two measures μ_1 and μ_2 on $A = \{0, 1, \dots, n\}$ so that convergence in the sense of this distance is equivalent to weak convergence of measures. The collection of all measures on A together with the function $L(\mu_1, \mu_2)$ forms the metric space $D(A)$. In accordance with [22] we introduce the following definition.

DEFINITION. A set T of measures on A *satisfies condition* (χ) if: (χ_1) the values $\mu(A)$, $\mu \in T$, are bounded, and (χ_2) given $\varepsilon > 0$, there exists a finite set k_2 of points such that $\mu(A \setminus k_\varepsilon) < \varepsilon$ for all $\mu \in T$.

In [22] it is proved that for the compactness of $T \subseteq D(A)$ it is necessary and sufficient that (χ) be satisfied. For $\{\pi_j(\nu)\}$, condition (χ) obviously implies condition (λ). Therefore, as a result of Theorem 3.1, we have the following theorem.

THEOREM 3.2. *In order that the stationary probabilities $\pi_j(\nu)$ depend continuously on ν for all $j \in A$ it is sufficient that the family $\{\pi_j(\nu)\}$ of distributions be compact in $D(A)$.*

Let us pass to the proof of Theorem 3.1. Let $\{\pi_j(\nu)\}$ satisfy condition (λ) at $\nu = 0$. We prove that $\pi_j(\nu)$ is right continuous at $\nu = 0$ for any $j \in A$ (the left continuity can be proved analogously). Take an arbitrary $\varepsilon > 0$. Condition (λ) implies the existence of a $\nu_0 > 0$ and a finite set $B^\varepsilon \subset A$ such that for any ν for which

$$0 \leqslant \nu \leqslant \nu_0 \qquad \text{(condition } a_1)$$

we have the inequality

$$\sum_{k\in A\setminus B^\varepsilon} \pi_k(\nu) < \varepsilon. \tag{3.4}$$

We prove that for any $j \in A$ there is a $\nu_1(j)$ such that for $0 < \nu < \nu_1(j)$

$$|\pi_j(\nu) - \pi_j(0)| < 10\varepsilon. \tag{3.5}$$

By the same token, we prove the continuity of the $\pi_j(\nu)$ at $\nu = 0$. The following inequality is satisfied for any t, i and j:

$$\begin{aligned} |\pi_j(\nu) - \pi_j(0)| &= |p_{ij}(t,\nu) - p_{ij}(t,0) + \pi_j(\nu) - p_{ij}(t,\nu) + p_{ij}(t,0) - \pi_j(0)| \\ &\leqslant |P_{ij}(t,\nu) - P_{ij}(t,0)| + |\pi_j(\nu) - p_{ij}(t,\nu)| + |p_{ij}(t,0) - \pi_j(0)|. \end{aligned} \tag{3.6}$$

From the ergodicity of the chains L^ν and L^0 for fixed i and j it follows that there is a $t_0(\nu)$ such that for

$$t > t_0(\nu) \qquad \text{(condition } a_2)$$

we have

$$\begin{gathered} |\pi_j(\nu) - p_{ij}(t,\nu)| < \varepsilon, \\ |p_{ij}(t,0) - \pi_j(0)| < \varepsilon. \end{gathered} \tag{3.7}$$

Consider the first term on the right side of (3.6). For any $T < t$ we have

$$\begin{aligned} |p_{ij}(t,\nu) - p_{ij}(t,0)| &= \Big| \sum_{k\in B^\varepsilon} p_{ik}(t-T,\nu) P_{kj}(T,\nu) \\ &+ \sum_{k\in A\setminus B^\varepsilon} p_{ij}(t-T,\nu) p_{kj}(T,\nu) - \sum_{k\in B^\varepsilon} p_{ik}(t-T,0) p_{kj}(T,0) \\ &- \sum_{k\in A\setminus B^\varepsilon} p_{ik}(t-T,0) p_{kj}(T,0) \Big| \leqslant \Big| \sum_{k\in B^\varepsilon} [p_{ik}(t-T,\nu) p_{kj}(T,\nu) \\ &- p_{ik}(t-T,0) p_{kj}(T,0)] \Big| + \sum_{k\in A\setminus B^\varepsilon} p_{ik}(t-T,\nu) + \sum_{k\in A\setminus B^\varepsilon} p_{ik}(t-T,0). \end{aligned} \tag{3.8}$$

From (3.4) and the ergodicity of L^ν and L^0 it follows that there is a $t_1(\nu)$ such that for

$$t - T > t_1(\nu) \qquad \text{(condition } a_3)$$

we have

$$\sum_{k\in A\setminus B^\varepsilon} p_{ik}(t-T,\ \nu)<2\varepsilon,$$
$$\sum_{k\in A\setminus B^\varepsilon} p_{ik}(t-T,0)<2\varepsilon. \tag{3.9}$$

Let us pass to the study of the first term on the right side of (3.8),

$$\begin{aligned}
&\Big|\sum_{k\in B^\varepsilon}[p_{ik}(t-T,\nu)p_{kj}(T,\nu)-p_{ik}(t-T,0)p_{kj}(T,0)]\Big|\\
&=\Big|\sum_{k\in B^\varepsilon}[p_{ik}(t-T,\nu)p_{ij}(T,\nu)-p_{ik}(t-T,\nu)(p_{ij}(T,\nu)-p_{kj}(T,\nu))\\
&-p_{ik}(t-T,0)p_{ij}(T,0)+p_{ik}(t-T,0)(p_{ij}(T,0)-p_{kj}(T,0))]\Big|\\
&\leqslant\Big|\sum_{k\in B^\varepsilon}p_{ik}(t-T,\nu)\,|p_{ij}(T,\nu)-p_{kj}(T,\nu)|\\
&+\sum_{k\in B^\varepsilon}p_{ik}(t-T,0)\,|p_{ij}(T,0)-p_{kj}(T,0)\Big|\\
&+\Big|p_{ij}(T,\nu)\sum_{k\in B^\varepsilon}p_{ik}(t-T,\nu)-p_{ij}(T,0)\sum_{k\in B^\varepsilon}p_{ik}(t-T,0)\Big|\\
&\leqslant\Big|p_{ij}(T,\nu)\sum_{k\in B^\varepsilon}p_{ik}(t-T,\nu)-p_{ij}(T,0)\sum_{k\in B^\varepsilon}p_{ik}(t-T,0)\Big|\\
&+\sum_{k\in B^\varepsilon}|p_{ij}(T,\nu)-p_{kj}(T,\nu)|+\sum_{k\in B^\varepsilon}|p_{ij}(T,0)-p_{kj}(T,0)|.
\end{aligned}\tag{3.10}$$

It follows from (3.9) that for $t - T > t_1(\nu)$ we have

$$1-\sum_{k\in B^\varepsilon}p_{ik}(t-T,\nu)<2\varepsilon,$$
$$1-\sum_{k\in B^\varepsilon}p_{ik}(t-T,0)<2\varepsilon. \tag{3.11}$$

Since $p_{ij}(T, \nu)$ is a continuous function of ν at zero for a fixed T, there is a $\nu_2 = \nu_2(T)$ such that for

$$0<\nu<\nu_2(T) \qquad \text{(condition } a_4\text{)}$$

we have

$$|p_{ij}(T,\nu)-p_{ij}(T,0)|<\varepsilon. \tag{3.12}$$

It follows from (3.11) and (3.12) that

$$\Big|p_{ij}(T,\nu)\sum_{k\in B^\varepsilon}p_{ik}(t-T,\nu)-p_{ij}(T,0)\sum_{k\in B^\varepsilon}p_{ik}(t-T,0)\Big|\leqslant 3\varepsilon. \tag{3.13}$$

The chain L^0 is ergodic; consequently there is a

$$T=T(B^\varepsilon) \qquad \text{(condition } a_5\text{)}$$

for which

$$\max_{k\in B^\varepsilon} |p_{ij}(T, 0) - p_{kj}(T, 0)| < \varepsilon/M, \tag{3.14}$$

$$\sum_{k\in B^\varepsilon} |p_{ij}(T, 0) - p_{kj}(T, 0)| < \varepsilon, \tag{3.15}$$

where M is equal to the number of terms in the sum (3.15). The continuity of $p_{kj}(T,\nu)$ at $\nu = 0$ for a fixed T implies the existence of $\nu_3(T)$ such that for

$$\nu < \nu_3(T) \qquad \text{(condition } a_6)$$

we have

$$\sup_{k\in B^\varepsilon, \nu<\nu_3(T)} |p_{kj}(T, \nu) - p_{kj}(T, 0)| < \varepsilon/M. \tag{3.16}$$

The inequalities (3.14) and (3.16) imply

$$\sup_{k\in B^\varepsilon, \nu<\nu_3(T)} |p_{ij}(T, \nu) - p_{kj}(T, \nu)| < 2\varepsilon/M. \tag{3.17}$$

Consequently,

$$\sum_{k\in B^\varepsilon} |p_{ij}(T, \nu) - p_{kj}(T, \nu)| < 2\varepsilon. \tag{3.18}$$

Comparing (3.9), (3.13), (3.15), and (3.18), we obtain that if t, T, and ν satisfy conditions a_1–a_6, then

$$|\pi_j(\nu) - \pi_j(0)| < 10\varepsilon. \tag{3.19}$$

The proof of the right continuity of $\pi_j(\nu)$ at $\nu = 0$ becomes complete if we show there exist T and $\nu_1 > 0$ such that for $0 < \nu < \nu_1$ there is a t depending on ν for which conditions $a_1 - a_6$ are satisfied. This can be shown easily in the following way. For the set B^ε we find $T = T(B^\varepsilon)$ (condition a_5), and for T we find numbers $\nu_2(T)$ and $\nu_3(T)$. Set

$$\nu_1(T) = \min\{\nu_0, \nu_2(T), \nu_3(T)\}.$$

Then for any $\nu < \nu_1(T)$ we take $t > \max\{t_0(\nu), T + t_1(\nu)\}$, which leads to the satisfaction of conditions a_1–a_6.

Now we prove that the continuity of $\pi_j(\nu)$ at $\nu = 0$ for any $j \in A$ implies the satisfaction of condition (λ) for $\{\pi_j(\nu)\}$ at $\nu = 0$. The chain L^0 is ergodic. Consequently, for any $\varepsilon > 0$ there is a finite set B^ε for which

$$\sum_{k\in A\setminus B^\varepsilon} \pi_k(0) < \varepsilon/2,$$

$$\sum_{k\in B^\varepsilon} \pi_k(0) > 1 - \varepsilon/2. \tag{3.20}$$

The continuity of $\pi_k(\nu)$ at $\nu = 0$ implies that there is a ν_0 such that for $|\nu| < \nu_0$ we have

$$\max_{k\in B^\varepsilon} |\pi_k(\nu) - \pi_k(0)| < \varepsilon/2M, \tag{3.21}$$

where M is the number of elements in the set B^{ε}. Therefore,

$$\sum_{k\in B^{\varepsilon}} |\pi_k(\nu) - \pi_k(0)| < \varepsilon/2, \tag{3.22}$$

$$\sum_{k\in A\setminus B^{\varepsilon}} \pi_k(\nu) < \varepsilon. \tag{3.23}$$

Consequently, $\{\pi_j(\nu)\}$ satisfies condition (λ) at $\nu = 0$. The theorem is proved.

§2. Sufficient conditions for the continuity of stationary probabilities

As in §1, let us consider a family $\{L^{\nu}\}$ of irreducible aperiodic Markov chains with transition probabilities $p_{ij}(1, \nu) = p_{ij}(\nu)$ continuously depending on ν for $\nu \in D \subset R^1$ (D is an open set).

The theorems of this section will be formulated in terms of "test" functions. Later, the continuity of stationary probabilities of random walks in $\mathbf{Z}_+^n$ will be studied by means of the results of the present section.

Assume that on the set $A = \{0, 1, \dots\}$ there are given two families $f^{\nu} = \{f_i^{\nu}\}$ and $g^{\nu} = \{g_i^{\nu}\}$ ($i \in A, \nu \in D$) of real functions, where

$$\inf_{i\in A,\,\nu\in D} f_i^{\nu} > 0, \qquad \inf_{i\in A,\,\nu\in D} g_i^{\nu} = \delta > 0.$$

THEOREM 3.3. *Assume that for some finite nonempty set $B \subset A$ the functions $\{f_i^{\nu}\}$ and $\{g_i^{\nu}\}$ satisfy the following conditions*:

1. $\sum_{j=0}^{\infty} p_{ij}(\nu) f_j^{\nu} - f_i^{\nu} < -g_i^{\nu} \quad i \notin B,\ \nu \in D,$

2. $\sup_{i\in B,\,\nu\in D} \sum_{j=0}^{\infty} p_{ij}(\nu) f_j^{\nu} = \lambda < \infty,$

3. $g_i^{\nu} \to \infty$ *as* $i \to \infty$ *uniformly in* $\nu \in D$.

Then the chains L^{ν} are ergodic for every $\nu \in D$, and the stationary probabilities $\pi_j(\nu)$ are continuous in ν for $\nu \in D$ and $j \in A$.

PROOF. The ergodicity of L^{ν} for every $\nu \in D$ follows from the hypotheses of Theorem 1.5. We define by induction

$$y_i^{n+1}(\nu) = \sum_{j=0}^{\infty} p_{ij}(1, \nu) y_j^n(\nu); \quad y_i^1(\nu) = y_i(\nu) = f_i^{\nu}.$$

It is obvious that $y_i^n(\nu) \geqslant 0$ for any natural numbers i and n, and any $\nu \in D$. We have

$$y_i^2(\nu) = \sum_{j=0}^{\infty} p_{ij}(1, \nu) f_j^{\nu} < f_i^{\nu} - g_i^{\nu} = y_i(\nu) - g_i^{\nu}, \quad i \notin B,$$

$$y_i^2(\nu) \leqslant \lambda, \quad i \in B,$$

$$y_i^3(\nu) = \sum_{j=0}^{\infty} p_{ij}(1, \nu) y_j^2(\nu) \leqslant \lambda \sum_{j\in B} p_{ij}(1, \nu) + \sum_{j\notin B} p_{ij}(1, \nu)(y_j(\nu) - g_j^{\nu}).$$

Write

$$\lambda_1 = \sup_{i \in B, \nu \in D} g_i^\nu; \quad p_{iB}(n, \nu) = \sum_{j \in B} p_{ij}(n, \nu).$$

After easy calculations we obtain

$$y_i^3(\nu) \leqslant (\lambda + \lambda_1) p_{iB}(1, \nu) + y_i^2(\nu) - \sum_{j=0}^{\infty} p_{ij}(1, \nu) g_j^\nu. \tag{3.24}$$

Moreover, the formula

$$y_i^n(\nu) \leqslant (\lambda + \lambda_1) p_{iB}(n-2, \nu) + y_i^{n-2}(\nu) - \sum_{j=0}^{\infty} p_{ij}(n-2, \nu) g_j^\nu \tag{3.25}$$

can be proved easily by induction.

We obtain from the recurrence relation (3.25) that

$$y_i^{n+2}(\nu) \leqslant y_i^2(\nu) + (\lambda + \lambda_1) \sum_{r=1}^{n} p_{iB}(r, \nu) - \sum_{r=1}^{n} \sum_{j=0}^{\infty} p_{ij}(r, \nu) g_j^\nu,$$

$$\frac{\sum_{r=1}^{n} \sum_{j=0}^{\infty} p_{ij}(r, \nu) g_j^\nu}{n} \leqslant \frac{y_i^2}{n} + (\lambda + \lambda_1) \frac{\sum_{r=1}^{n} p_{iB}^r}{n} - \frac{y_i^{n+2}(\nu)}{n}$$

$$< \frac{y_i^2(\nu)}{n} + \lambda + \lambda_1 < y_i^2(\nu) + \lambda + \lambda_1. \tag{3.26}$$

Take $i \in B$. Then

$$\sum_{j=0}^{\infty} \left(\frac{\sum_{r=1}^{n} p_{ij}(r, \nu)}{n} g_i^\nu \right) < y_i^2(\nu) + \lambda + \lambda_1 < 2\lambda + \lambda_1. \tag{3.27}$$

Take an arbitrary $M > 0$. Consider the set $B^M = \{i\colon \min_{\nu \in D} g^\nu(i) < M\}$. Since $g_i^\nu \to \infty$ uniformly in $\nu \in D$ as $i \to \infty$, the set B^M is finite. Besides, if $j \in A \setminus B^M$, then for any $\nu \in D$ we have $g_j^\nu \geqslant M$. It follows from (3.27) that

$$\sum_{j \in A \setminus B^M} \frac{\sum_{r=1}^{n} p_{ij}(r, \nu)}{n} g_j^\nu < 2\lambda + \lambda_1.$$

From this we obtain

$$\sum_{j \in A \setminus B^M} \frac{\sum_{r=1}^{n} p_{ij}(r, \nu)}{n} < \frac{2\lambda + \lambda_1}{M}. \tag{3.28}$$

From (3.28) it follows that

$$\sum_{j \in B^M} \frac{\sum_{r=1}^{n} p_{ij}(r, \nu)}{n} > 1 - \frac{2\lambda + \lambda_1}{M}. \tag{3.29}$$

We have

$$\pi_j(\nu) = \lim_{n \to \infty} \frac{\sum_{r=1}^{n} p_{ij}^{r}(\nu)}{n}.$$

Therefore, from the finiteness of B^M and (3.29) it follows that

$$\sum_{j \in B^M} \pi_j(\nu) > 1 - \frac{2\lambda + \lambda_1}{M},$$
$$\sum_{j \in A \setminus B^M} \pi_j(\nu) < \frac{2\lambda + \lambda_1}{M}. \tag{3.30}$$

Since the number M can be chosen arbitrarily large and we can construct the set B^M for it, (3.30) implies the compactness of the family $\{\pi_j(\nu)\}$ of distributions for $\nu \in D$, which implies the continuity of $\pi_j(\nu)$ for any $j \in A$ and $\nu \in D$ by Theorem 3.2. The theorem is proved.

THEOREM 3.4. *Assume that the following conditions are satisfied for some $\delta > 0$, some $\gamma > 1$, and a finite nonempty set $B \subset A$:*

1. $\sum_{j=0}^{\infty} p_{ij}(\nu) f_j^{\nu} - f_i^{\nu} < -\delta, \quad i \notin B, \ \nu \in D.$

2. $\sup_{i \in B, \nu \in D} \sum_{j=0}^{\infty} p_{ij}(\nu)(f_j^{\nu})^{\gamma} = \lambda_{\gamma} < \infty.$

3. $\sup_{i \in A, \nu \in D} \sum_{j=0}^{\infty} p_{ij}(\nu) |f_j^{\nu} - f_i^{\nu}|^{\gamma} = c_{\gamma} < \infty.$

4. $f_i^{\nu} \to \infty$ *uniformly in $\nu \in D$ as $i \to \infty$.*

Then the chains L^{ν} are ergodic for every $\nu \in D$, and the stationary probabilities $\pi_j(\nu)$ are continuous in ν for $\nu \in D$ and $j \in A$.

PROOF. For some $\gamma > 1$ let condition 3 of Theorem 3.4 be satisfied. Then for any γ_0 such that $1 \leqslant \gamma_0 < \gamma$ we have

$$\sup_{i \in A, \nu \in D} \sum_{j=0}^{\infty} p_{ij}(\nu) |f_j^{\nu} - f_i^{\nu}|^{\gamma_0} < c_{\gamma} + 1 < \infty.$$

Therefore, without loss of generality we may assume that $1 < \gamma \leqslant 2$. For any such γ and any $y, x > 0$ we prove the auxiliary inequality

$$y^{\gamma} - x^{\gamma} \leqslant |y - x|^{\gamma} + x^{\gamma-1}\gamma(y - x). \tag{3.31}$$

Set $z = y/x$. Inequality (3.31) can be rewritten as

$$z^{\gamma} - 1 - |z - 1|^{\gamma} - \gamma(z - 1) \leqslant 0. \tag{3.32}$$

For the proof of (3.32) we consider two cases.

1. Let $z \geqslant 1$. Then for $z = 1$ the left side of (3.32) is equal to zero, and for $z > 1$ we have

$$\frac{d(z^\gamma - 1 - (z-1)^\gamma - \gamma(z-1))}{dz} = \gamma[z^{\gamma-1} - (z-1)^{\gamma-1} - 1] < 0.$$

Consequently, (3.32) is satisfied for $z \geqslant 1$.

2. Let $z \leqslant 1$. Then inequality (3.32) turns into an equality for $z = 1$, and for $z < 1$ we have

$$\frac{d(z^\gamma - 1 - (1-z)^\gamma - \gamma(z-1))}{dz} = \gamma[z^{\gamma-1} + (1-z)^{\gamma-1} - 1] > 0.$$

Consequently, (3.32) is satisfied for $z < 1$, and together with it inequality (3.31) is also satisfied for any $1 \leqslant \gamma \leqslant 2$ and $y, x > 0$.

Let us use (3.31) to estimate $\sum_{j=0}^\infty p_{ij}(\nu)[(f_j^\nu)^\gamma - (f_i^\nu)^\gamma]$ for $i \notin B$. We have

$$\sum_{j=0}^\infty p_{ij}(\nu)[(f_j^\nu)^\gamma - (f_i^\nu)^\gamma] \leqslant \sum_{j=0}^\infty p_{ij}(\nu)[|f_j^\nu - f_i^\nu|^\gamma + (f_i^\nu)^{\gamma-1}\gamma(f_j^\nu - f_i^\nu)]$$

$$\leqslant \sum_{j=0}^\infty p_{ij}(\nu)|f_j^\nu - f_i^\nu|^\gamma + \gamma(f_i^\nu)^{\gamma-1}\sum_{j=0}^\infty p_{ij}(\nu)(f_j^\nu - f_i^\nu).$$

Taking account of conditions 1 and 3 in Theorem 3.4, we finally obtain the estimate

$$\sum_{j=0}^\infty p_{ij}(\nu)[(f_j^\nu)^\gamma - (f_i^\nu)^\gamma] \leqslant c_\gamma - \gamma(f_i^\nu)^{\gamma-1}\delta. \tag{3.33}$$

If for some family $\{f_i^\nu\}$ of functions the hypotheses of Theorem 3.4 are satisfied, then they will also be satisfied for the family $\{f_i^\nu + r\} = \{\tilde{f}_i^\nu\}$, where $r > 0$ is arbitrary. (The second hypothesis of the theorem will be satisfied with another constant $\tilde{\lambda}_\nu < \infty$.) Therefore, without loss of generality we may assume that $\gamma(f_i^\nu)^{\gamma-1}\delta - c_\gamma > \sigma > 0$ for some σ and any $i \in A$, $\nu \in D$. Set $\tilde{f}_i^\nu = (f_i^\nu)^\gamma$ and $\tilde{g}_i^\nu = \gamma(f_i^\nu)^{\gamma-1}\delta - c_\gamma$. Using (3.33) and condition 2 in the theorem, we obtain

$$\sum_{j=0}^\infty p_{ij}(\nu)\tilde{f}_j^\nu - \tilde{f}_i^\nu < -\tilde{g}_i^\nu, \quad i \notin B, \ \nu \in D,$$

$$\sum_{j=0}^\infty p_{ij}(\nu)\tilde{f}_j^\nu < \tilde{\lambda}_\nu < \infty, \quad i \in B, \ \nu \in D.$$

Since $\gamma > 1$, the fact that the f_i^ν tend to infinity uniformly as $i \to \infty$ implies the same for $g_i^\nu = \gamma(f_i^\nu)^{\gamma-1}\delta - c_\gamma$. Hence, for the functions $\{\tilde{f}_i^\nu\}$ and $\{\tilde{g}_i^\nu\}$ all hypotheses of Theorem 3.3 are satisfied. Consequently, the $\pi_j(\nu)$ are continuous in ν for $\nu \in D$ and $j \in A$. The theorem is proved.

REMARK. Let $\xi_0, \xi_1, \ldots$ be the sequence of random variables corresponding to the Markov chain L. Let C be a set, $C \subset A = \{0, 1, \ldots\}$. For $i \in C$ introduce

$$f_{iC}^n = p\{\xi_i \notin C, \ldots, \xi_{n-1} \notin C, \ \xi_n \in C / \xi_0 = i\}.$$

Then the mean first passage time of the process $\{\xi_i\}$ to the set C under the condition $\xi_0 = i$ is defined by

$$Mt = \sum_{n=1}^{\infty} n f_{iC}^n .$$

For any $\gamma > 0$ we can define the quantity

$$Mt^\gamma = \sum_{n=1}^{\infty} n^\gamma f_{iC}^n .$$

Following [18], the property that Mt^γ is finite will be called *γ-recurrence* of the set C. As follows from results of [18], the hypotheses of Theorem 3.4 guarantee the γ-recurrence of B, uniformly in $\nu \in D$. Moreover, it can be shown that the uniform γ-recurrence of the family of chains $\{L^\nu\}$ with $\gamma > 1$ implies the continuity of the stationary probabilities. In this way, we have outlined still another method of proof of Theorem 3.4.

On the set $A = \{0, 1, \ldots\}$ let a family of integral-valued, positive, uniformly bounded functions $k^\nu = \{k_i^\nu\}$ be given for which

$$\sup_{i \in A, \nu \in D} k_i^\nu = b < \infty .$$

Concerning the function $\{f_i^\nu\}$ already introduced, we assume the following condition to be satisfied.

Boundedness Condition. *There is a $d > 0$ such that $\sup_{\nu \in D} |f_i^\nu - f_j^\nu| > d$ implies that $p_{ij}(\nu) = 0$.*

Theorem 3.5. *Assume that the inequalities*

$$\sum_{j=0}^{\infty} p_{ij} (k_i^\nu, \nu) f_j^\nu - f_i^\nu < -\varepsilon \tag{3.34}$$

are satisfied for some $\varepsilon > 0$, all $\nu \in D$, and all i except some finite nonempty set B. Then the chains L^ν are ergodic for every $\nu \in D$, and the stationary probabilities $\pi_j(\nu)$ are continuous in ν for $\nu \in D$ and $j \in A$.

We precede the proof of the theorem by two lemmas. Let $\{\xi_i^\nu\}$ be the sequence of random variables corresponding to L^ν. We have

$$f_{ij}^n(\nu) = P\{\xi_1^\nu \neq j, \ldots, \xi_{n-1}^\nu \neq j, \xi_n^\nu = j / \xi_0^\nu = i\}.$$

Lemma 3.2. *The functions $f_{ij}^n(\nu)$ are continuous in ν $(\nu \in D)$ for any natural number n and any $i, j \in A$.*

Proof. We prove the lemma by induction. For $n = 1$ the function $f_{ij}^1(\nu) = p_{ij}(\nu)$ is continuous in ν for any i and j. Assume that $f_{ij}^n(\nu)$ is continuous in ν for any $i,j \in A$. Use the formula

$$f_{ij}^{n+1}(\nu) = \sum_{k=0}^{\infty} p_{ik}(\nu) f_{kj}^n(\nu) - p_{ij}(\nu) f_{jj}^n(\nu).$$

As in Lemma 3.1 of the preceding section, we can prove the uniform convergence of the series $\sum_{k=0}^{\infty} p_{ik}(\nu) f_{kj}^n(\nu)$, which consists of continuous functions. This implies the continuity of $f_{ij}^{n+1}(\nu)$.

LEMMA 3.3. *Let the hypotheses of Theorem* 3.5 *be satisfied. Then for any points* $i_0 \in A$ *and* $\nu_0 \in D$ *chosen beforehand there exist functions* $\{\tilde{k}_i^\nu\}$ *and* $\{\tilde{f}_i^\nu\}$ *such that* $\sup_{i\in A, \nu\in D} \tilde{k}_i^\nu = \tilde{b} < \infty$, *and the boundedness condition is satisfied for the functions* $\{\tilde{f}_i^\nu\}$ *with a constant* $\tilde{d} > 0$, *and in place of* (3.34) *the inequality*

$$\sum_{j=0}^{\infty} p_{ij}(\tilde{k}_i^\nu, \nu)\,\tilde{f}_j^\nu - \tilde{f}_i^\nu < -\varepsilon_1$$

holds for some $\varepsilon_1 > 0$, *all* $\nu \in D$, *and all* $i \in A$ *except* $i = i_0$.

PROOF. Without loss of generality we may set $i_0 = 0$. From the assumption that L^{ν_0} has a single essential class of states, it follows for any point $j \in B$ that there exist a positive integer $r_j(\nu_0)$ and $\varepsilon_1 > 0$ such that $p_{j0}(r_j(\nu_0), \nu_0) > \varepsilon_1$. Since $p_{ij}(n, \nu)$ is a continuous function of ν ($\nu \in D$, and i, j and n are arbitrary), there exists a neighborhood D_j of ν_0 such that $p_{j0}(r_j(\nu_0), \nu) > \varepsilon_2 = \varepsilon_1/2$ for any $\nu \in \tilde{D}_j$. Set

$$\tilde{D} = \bigcap_{j\in B} \tilde{D}_j,$$

$$\tilde{k}_j^\nu = \begin{cases} k_j^\nu, & j\notin B,\ \nu\in\tilde{D}, \\ r_j^{\nu_0}, & j\in B,\ \nu\in\tilde{D}, \end{cases}$$

$$\tilde{f}_j^\nu = \begin{cases} f_j^\nu, & j\neq 0,\ \nu\in\tilde{D}, \\ f_0^\nu - 2\dfrac{d\tilde{b}}{\varepsilon_2}, & j=0,\ \nu\in\tilde{D}, \end{cases}$$

where $\tilde{b}$ is defined in the following way:

$$\sup_{j\in A,\,\nu\in D} \tilde{k}_j^\nu = \tilde{b} < \infty, \quad \tilde{d} = d + 2\frac{d\tilde{b}}{\varepsilon_2}.$$

For any $i \notin B \cup 0$ we have

$$\sum_{j\in A} p_{ij}(\tilde{k}_i^\nu, \nu)\,\tilde{f}_j^\nu - \tilde{f}_i^\nu < -\varepsilon \quad (\nu\in\tilde{D}). \tag{3.35}$$

For $i \in B \setminus 0$ and $\nu \in \tilde{D}$ we have

$$\sum_{j=0}^{\infty} p_{ij}(\tilde{k}_i^\nu, \nu)\,\tilde{f}_j^\nu - \tilde{f}_i^\nu = \sum_{j=0}^{\infty} p_{ij}(\tilde{k}_i^\nu, \nu)\, f_j^\nu - p_{i0}(\tilde{k}_i^\nu, \nu)\,\frac{2d\tilde{b}}{\varepsilon_2}$$
$$- f_i^\nu < f_i^\nu + d\tilde{k}_i^\nu - 2d\tilde{b} - f_i^\nu < -d\tilde{b}. \tag{3.36}$$

Hence, if we set $\tilde{\varepsilon} = \min(\varepsilon, d\tilde{b})$, we finally obtain

$$\sum_{j=0}^{\infty} p_{ij}(\tilde{k}_i^\nu, \nu)\,\tilde{f}_j^\nu - \tilde{f}_i^\nu < -\tilde{\varepsilon} \quad (\nu\in\tilde{D},\ i\neq 0). \tag{3.37}$$

The proof of the lemma is complete.

PROOF OF THEOREM 3.5. The ergodicity of the chains L^ν ($\nu \in D$) follows from Theorem 1.5. Let us fix any point $i_0 \in A$. Without loss of generality, let $i_0 = 0$. We prove that $\pi_0(\nu)$ is a continuous function of ν for $\nu \in D$. Take an arbitrary point $\nu_0 \in D$. It follows from Lemma 3.3 that we can correct the functions f_i^ν and k_i^ν so that for some neighborhood $\tilde{D}$ of ν_0 we have (3.37). We may assume that $\tilde{f}_0^\nu < f_i^\nu$ for any $i \in A$ and $\nu \in \tilde{D}$, since otherwise this can be achieved easily by decreasing the values of $\tilde{f}_0^\nu$; in this case the inequalities (3.37) are not violated. We introduce the mean time

$$m_0(\nu) = \sum_{n=1}^{\infty} n f_{00}^n(\nu)$$

of entrance of L^ν to the null state. It follows from the ergodicity of the chains L^ν ($\nu \in D$) that $m_0(\nu)$ is finite, and

$$\pi_0(\nu) = 1/m_0(\nu).$$

We show that $m_0(\nu)$ is continuous in ν for $\nu \in D$. As in Theorem 1.6, from the sequence of random variables $\xi_0^\nu = 0, \xi_1^\nu, \ldots$ corresponding to L^ν we form a random sequence $\{S_n^\nu\}$ by setting $S_n^\nu = f^\nu\{\xi_n^\nu\}$. From the sequence $\{\xi_1^\nu\}$ we also form an integral-valued sequence $\{N_1^\nu\}$ by setting $N_0^\nu = \tilde{k}^\nu(\xi_0^\nu)$ and $N_i^\nu = N_{i-1}^\nu + \tilde{k}^\nu(\xi_{i-1}^\nu)$. It follows from the uniform boundedness of the functions $\tilde{k}_i^\nu$ ($i \in A, \nu \in D$) that $1 \leqslant N_{i+1}^\nu - N_i^\nu \leqslant \tilde{b}$ for any $i \in A$ and $\nu \in D$ with probability 1. It follows from (3.37) that

$$M\left(S_{N_i}^\nu / S_{N_{i-1}}^\nu > \tilde{f}_0^\nu\right) < S_{N_{i-1}}^\nu - \varepsilon \tag{3.38}$$

with probability 1. We have

$$\begin{aligned} f_{00}^n(\nu) &= P\{\xi_1^\nu \neq 0, \ldots, \xi_{n-1}^\nu \neq 0, \xi_n^\nu = 0/\xi_0^\nu = 0\} \\ &= P\{S_1^\nu > \tilde{f}_0^\nu, \ldots, S_{n-1}^\nu > \tilde{f}_0^\nu, S_n^\nu = \tilde{f}_0^\nu / S_0^\nu = \tilde{f}_0^\nu\}. \end{aligned}$$

Therefore, taking account of (3.38) and applying Lemma 1.2, we obtain the following estimate for $f_{00}^n(\nu)$:

$$f_{00}^n(\nu) < c \exp\{-\delta n\}, \quad n \in A, \ \nu \in \tilde{D}, \tag{3.39}$$

where $c, \delta > 0$ are constants not depending on ν. Taking account of (3.39), we conclude that the series $\sum_{n=1}^\infty n f_{00}^n(\nu)$, which consists of continuous functions (cf. Lemma 3.2), converges uniformly in ν for $\nu \in \tilde{D}$. Therefore, the sum $m_0(\nu)$ of the series is a continuous function of ν for $\nu \in D$, which in turn implies the continuity of $\pi_0(\nu)$ in ν for $\nu \in \tilde{D}$. It remains to note that $i = 0$ and $\nu_0 \in D$ were chosen arbitrarily. Therefore, the $\pi_j(\nu)$ are continuous functions of the parameter ν for $\nu \in D$ and any $j \in A$. The proof of the theorem is finished.

§3. Continuity of random walks in $\mathbf{Z}_+^n$

Consider a family $\{L^\nu\}$ of homogeneous irreducible aperiodic Markov chains with discrete time and set of states $\mathbf{Z}_+^n = \{(z_1, \ldots, z_n): z_i \geqslant 0, \text{integral}\}$ ($\nu \in D$, where D is an open subset of the real line). $p_{\alpha\beta}(t, \nu)$ ($\alpha, \beta \in \mathbf{Z}_+^n$) is the probability of the transition of L^ν from the point α to the point β in t steps.

Concerning the family $\{L^\nu\}$ of random walks, we assume that the homogeneity condition and the boundedness of jumps hold uniformly in $\nu \in D$. Let B^Λ_{ct} be the sets introduced in §1 of Chapter II.

HOMOGENEITY CONDITION. *There is a $c > 0$ such that for any Λ, any vector $a = (a_1, \ldots, a_n)$ with $a_i \geq 0$, $1 \leq i \leq n$, and $a_j = 0$ for $j \notin \Lambda$, and all $\alpha \in B^\Lambda_{cc} \cap \mathbf{Z}^n_+$ we have*

$$p_{\alpha\beta}(\nu) = p_{\alpha+a,\beta+a}(\nu)$$

for any $\beta \in \mathbf{Z}^n_+$ and $\nu \in D$.

BOUNDEDNESS OF JUMPS. *For any α the number of β's such that $\sup_{\nu\in D} p_{\alpha\beta}(\nu) > 0$ is finite.*

By the homogeneity condition this is equivalent to the following: *There exists $d > 0$ such that $\|\alpha - \beta\| > d$ implies $p_{\alpha\beta}(\nu) = 0$ for any ν.*

As before, we shall assume that the $p_{\alpha\beta}(1, \nu)$ are continuous functions of ν for any $\alpha, \beta \in \mathbf{Z}^n_+$ and $\nu \in D$. For every chain L^ν let us construct a vector field V^ν by the method indicated in §1 of Chapter II. Then we obtain a family $\{V^\nu\}$ of vector fields.

We shall say that *the family $\{V^\nu\}$ $(\nu \in \tilde{D} \subset D)$ satisfies condition* $\tilde{\mathbf{B}}$ if for some δ, b, and $p > 0$ there is a function $f(\alpha)$ $(\alpha \in R^n_+)$ which satisfies the following conditions:

1. $f(\alpha) > 0, \quad \alpha \in R^n_+$.
2. $f(\alpha) - f(\beta) \leqslant b\|\alpha - \beta\|$ for any $\alpha, \beta \in R^n_+$.
3. For any Λ either all the $L^\Lambda(\nu)$ $(\nu \in \tilde{D})$ or none of them are ergodic.
4. For any Λ such that $L^\Lambda(\nu)$ is ergodic and for $\Lambda = \{1, \ldots, n\}$ and all $\alpha \in B^\Lambda \cap B^\Lambda_{pp}$ we have

$$\sup_{\nu\in\tilde{D}} (f(\alpha + v^\nu(\alpha)) - f(\alpha)) < -\delta.$$

THEOREM 3.6. *If there exists a set $U \subset D$ such that $\{V^\nu\}$ $(\nu \in U)$ satisfies condition $\tilde{\mathbf{B}}$, then for all $\nu \in U$ the chains L^ν are ergodic, and the stationary probabilities $\pi_\alpha(\nu)$ are continuous in ν for any $\alpha \in \mathbf{Z}^n_+$ and $\nu \in U$.*

PROOF. Assume that there exist a set $U \subset D$ and a function $f(\alpha)$, $\alpha \in R^n_+$, such that condition $\tilde{\mathbf{B}}$ is satisfied. Set $f^\nu \equiv f$, i.e. set $f^\nu(\alpha) = f(\alpha)$ for every $\alpha \in \mathbf{Z}^n_+$ and $\nu \in U$. It follows from the proof of Theorem 3.1 that for any $\nu \in U$ there is a function $m^\nu(\alpha)$ such that

$$\sup_{\alpha\in\mathbf{Z}^n_+} m^\nu(\alpha) = m^\nu < \infty$$

and for all $\alpha \in \mathbf{Z}^n_+$, except some finite set C^ν,

$$\sum_{\beta\in\mathbf{Z}^n_+} p_{\alpha\beta}(m(\alpha), \nu) f^\nu_\beta - f^\nu_\alpha < -\varepsilon_1(\nu) \qquad (3.40)$$

for some $\varepsilon_1(\nu) > 0$. From the method of proof of Theorem 2.1 it follows that

$$\sup_{\nu \in U} m^\nu < \infty,$$
$$\inf_{\nu \in U} \varepsilon_1(\nu) > 0, \tag{3.41}$$

and $\bigcup_{\nu \in U} C^\nu$ is a finite set.

From (3.40) and (3.41) we conclude that all hypotheses of Theorem 3.5 are satisfied. Consequently, all chains L^ν are ergodic for $\nu \in U$, and the stationary probabilities $\pi_\alpha(\nu)$ are continuous in ν for $\nu \in U$ and any $\alpha \in \mathbf{Z}_+^n$. The theorem is proved.

For a random walk L in $\mathbf{Z}_+^n$, where $n \leqslant 3$, in §2 of Chapter II a method was given for constructing the function $f(\alpha)$ satisfying condition B, which leads to the formulation of ergodicity conditions in terms of random walks of lower dimensions. Now we prove a theorem showing that the satisfaction of these ergodicity conditions for the chain L^{ν_0} guarantees the continuity of the stationary probabilities of $\{L^\nu\}$ in some neighborhood of ν_0. We formulate the theorem for random walks in $\mathbf{Z}_+^3$. (This can be done analogously for $\mathbf{Z}_+^1$ or $\mathbf{Z}_+^2$.)

THEOREM 3.7. *Assume that the Markov chain L^{ν_0}, where $\nu_0 \in D$, satisfies the hypotheses of Theorem 2.4 which guarantee the ergodicity of L^{ν_0}. Then there exists a neighborhood $U \subset D$ of the point ν_0 such that for all $\nu \in U$ the chains L^ν are ergodic, and the stationary probabilities $\pi_\alpha(\nu)$ are continuous in ν for any $\alpha \in \mathbf{Z}_+^3$ and $\nu \in U$.*

PROOF. It follows from the proof of Theorem 2.4 that there exists a function $f(\alpha)$, $\alpha \in R_+^3$, which satisfies condition B introduced in §1 of Chapter II. The function $f(\alpha)$ satisfies the following conditions:

1. $f(\alpha) > 0, \quad \alpha \in R_+^3.$
2. $|f(\alpha) - f(\beta)| \leqslant b \| \alpha - \beta \| \quad \alpha, \beta \in R_+^3,$

where $b > 0$ is a constant. Condition B guarantees the existence of a function $m(\alpha)$, $\alpha \in \mathbf{Z}_+^3$, with values in the set of natural numbers such that

$$\sup_{\alpha \in Z_+^3} m(\alpha) = m < \infty$$

and for all $\alpha \in \mathbf{Z}_+^3$, except some finite set R,

$$\sum_{\beta \in Z_+^3} p_{\alpha\beta}(m(\alpha), \nu_0) f_\beta - f_\alpha < -\varepsilon \tag{3.42}$$

for some $\varepsilon > 0$. From the homogeneity condition and the boundedness of jumps for the family of random walks $\{L^\nu\}$ it follows that $p_{\alpha\beta}(t, \nu)$ is a continuous function of ν for $\nu \in D$ uniformly in $\alpha, \beta \in \mathbf{Z}_+^3$ (t is an arbitrarily given natural number). Therefore, taking account of properties 1 and 2 of the function $f(\alpha)$ we conclude that there is a neighborhood U of ν_0 such that for all $\nu \in U$ and $\alpha \in \mathbf{Z}^3_+ \setminus B$

$$\sum_{\beta \in Z^3_+} p_{\alpha\beta}(m(\alpha), \nu) f_\beta - f_\alpha < -\varepsilon/2. \tag{3.43}$$

From the properties of $f(\alpha)$ and the uniform boundedness of the jumps of the random walks in $\{L^\nu\}$ it follows that there is a $d > 0$ such that $\sup_{\nu \in D}|f_j^\nu - f_i^\nu| > d$ implies that $p_{ij}(\nu) = 0$. Hence all hypotheses of Theorem 3.5 are satisfied. Consequently, the chains L^ν are ergodic for all $\nu \in U$, and the stationary probabilities $\pi_\alpha(\nu)$ are continuous in ν for $\nu \in U$ and any $\alpha \in \mathbf{Z}^3_+$. The theorem is proved.

Chapter IV. Analyticity of a family of countable Markov chains and random walks

§1. The fundamental analyticity theorem for a family of countable Markov chains

Consider a family of homogeneous irreducible aperiodic Markov chains $\{L^\nu\}$ with discrete time and countable state set $A = \{0, 1, \dots\}$ and $\nu \in D$, where D is some neighborhood of zero on the real line. Let $\mathfrak{X}(A, \Sigma)$ be the Banach space of countably additive real measures on (A, Σ) with norm equal to the total variation (Σ is the σ-algebra of all subsets of A). It is easy to see that $\mathfrak{X}(A, \Sigma) \equiv l_1(A)$.

Denote by $B(\mathfrak{X})$ the Banach space of bounded linear operators on $\mathfrak{X}$. Any Markov chain L^ν on A defines an operator $P_\nu \in B(\mathfrak{X})$ with norm equal to 1.

DEFINITION. A set $M \subset \mathfrak{X}(A, \Sigma)$ is called a *set of uniform convergence* for the operator $P \in B(\mathfrak{X})$ if $PM \subset M$ and there exist a function $\varphi(n)$, $n = 1, 2, \dots,$ such that

$$\varphi = \sum_{n=1}^{\infty} \varphi(n) < \infty \tag{4.1}$$

and an element $y \in M$ such that

$$\| P^n x - y \| < \varphi(n) \tag{4.2}$$

for all n and $x \in M$.

THEOREM 4.1. *Let $P_\nu = P(\nu)$ depend analytically on ν as a function with values in the Banach algebra $B(\mathfrak{X})$ of operators, and assume that the following conditions are satisfied.*

1. *For the operator P_0 there exist two sets M_1 and M_2 of uniform convergence such that $M_1 \subset M_2$ and $\inf_{x \in M_2}\|x\| > 0$.*
2. *There is a $\nu_0 > 0$ such that $P_\nu x \in M_2$ for all $|\nu| < \nu_0$ and any $x \in M_1$.*
3. *There is a $\nu_1 > 0$ such that*

$$x_1 + \frac{(P_\nu - P_0) x_2}{\| P_\nu - P_0 \|} \in M_2$$

for $|\nu| < \nu_1$ and any $x_1, x_2 \in M_1$.

Then there is a $\nu_2 > 0$ such that for $|\nu| < \nu_2$ and $x \in M_1$ the limit $\lim_{n\to\infty} P^n(\nu)x = r(\nu)$ exists and depends analytically on ν.

PROOF. For any $B \in \mathfrak{X}(A, \Sigma)$ and $G \in B(\mathfrak{X})$ we write

$$\| G \|_B = \sup_{0 \neq x \in B} \frac{\| Gx \|}{\| x \|}.$$

It follows from the hypotheses of the theorem that there exist a function $\varphi(n)$ and an element $y \in M_2$ such that conditions (4.1) and (4.2) are satisfied for any $x \in M_2$.

LEMMA 4.1. *Under the hypotheses of Theorem* 4.1 *there is a constant* $c > 0$ *such that*

$$\| P_0^n (P_\nu - P_0) \|_{M_1} \leqslant c\varphi(n) \| P_\nu - P_0 \|. \tag{4.3}$$

PROOF. Take an arbitrary $x \in M_1$. Set $P_\nu x = z_1$ and $P_0 x = z_2$. We have

$$\begin{aligned} \| P_0^n (P_\nu - P_0) x \| &= \| P_0^n (z_1 - z_2) \| = \| P_0^n (z_2 \| P_\nu - P_0 \| - z_2 \| P_\nu \\ &- P_0 \| + (z_1 - z_2)) \| \leqslant \| P_\nu - P_0 \| \, \| P_0^n (z_2 - y) \| + \| P_\nu - P_0 \| \\ &\times \left\| P_0^n \left(z_2 + \frac{z_2 - z_1}{\| P_\nu - P_0 \|} - y \right) \right\| \leqslant \| P_\nu - P_0 \| \varphi(n) + \| P_\nu - P_0 \| \varphi(n) \\ &= 2 \cdot \varphi(n) \| P_\nu - P_0 \|. \end{aligned} \tag{4.4}$$

In the proof of (4.4) we have used the fact that, by assumption, $z_2 = P_0 x$ and $(z_2 - z_1)/\|P_\nu - P_0\|$ belong to the set M_2 of uniform convergence. Moreover, using (4.4), we have

$$\begin{aligned} \| P_0^n (P_\nu - P_0) \|_{M_1} &= \sup_{x \in M_1} \frac{\| P_0^n (P_\nu - P_0) x \|}{\| x \|} \\ &\leqslant \frac{2\varphi(n) \| P_\nu - P_0 \|}{\| x \|} \leqslant c\varphi(n) \| P_\nu - P_0 \|. \end{aligned}$$

The lemma is proved.

Let us continue the proof of the theorem. Set

$$Q = Q(\nu) = P_\nu - P_0. \tag{4.5}$$

Write

$$Q^{(k, i_1, i_2, \ldots, i_k)} = P_0^{i_1} Q P_0^{i_2} Q \ . \ . \ . P_0^{i_k} Q,$$

where $k \geqslant 1$, $j_j \geqslant 0$ $(j = 1, 2, \ldots, k)$, δ is the $(k + 1)$-tuple $(k, i_1, \ldots, i_k)$, and $Q^{\varnothing} = 1$. Let $r_0 = P_0^\infty x = \lim_{n\to\infty} P_0^n(x)$ $(x \in M_2)$. The existence and uniqueness of r_0 obviously follows from conditions (4.1) and (4.2) for the set M_2. We prove that $r_\nu = P_\nu^\infty x = \lim_{n\to\infty} P_\nu^n x$ exists for all $x \in M_1$, is unique, and can be represented in the form

$$r_\nu = \sum_\delta Q^\delta r_0 = \sum_{\substack{k \geqslant 1 \\ i_1, i_2, \ldots, i_k \geqslant 0}} Q^{(k, i_1, i_2, \ldots, i_k)} \cdot r_0 + r_0. \tag{4.6}$$

The latter series converges absolutely in $\mathfrak{X}(A, \Sigma)$ (r_ν does not necessarily belong to M_2). It follows from Lemma 4.1 that

$$\| P_0^n Q \|_{M_1} \leqslant c\varphi(n) \| P_\nu - P_0 \|. \tag{4.7}$$

Therefore, the series (4.6) is dominated by the numerical series

$$\sum_{\substack{k \geqslant 1 \\ i_1, i_2, \dots, i_k}} \varphi(i_1) \cdot \ldots \cdot \varphi(i_k) c^k \| P_\nu - P_0 \|^k + 1 = \sum_{k=0}^{\infty} (\| P_\nu - P_0 \| \cdot c\varphi)^k. \tag{4.8}$$

The series (4.8) is convergent, provided that $\|P_\nu - P_0\| < 1/c\varphi$. Consequently, the series (4.6) converges absolutely. Now we prove equality (4.6). For this we estimate the difference

$$\Big\| \sum_\delta Q^\delta P_0^\infty - (P_0 + Q)^n \Big\|_{M_1}.$$

The following equality is obvious:

$$(P_0 + Q)^n = P_0^n + P_0^{n-1} Q + P_0^{n-2} Q P_0 + \ldots + P_0^{n-k} Q P_0^{k+1} + \ldots + Q P^{n-1} + \ldots. \tag{4.9}$$

Therefore

$$\Big\| \sum_\delta Q^\delta P_0^\infty - (P_0 + Q)^n \Big\|_{M_1} \leqslant \sum_{\substack{k \geqslant 1 \\ i_1 + i_2 + \dots + i_k < n/2}} \| Q^\delta \| \| P_0^\infty - P_0^{n - i_1 - \dots - i_k} \|$$

$$+ \sum_{\substack{k \geqslant 1 \\ i_1 + i_2 + \dots + i_k \geqslant n/2}} \| Q^\delta \| \| P_0^\infty \| + \sum_{\substack{k \geqslant 1 \\ n > i_1 + \dots + i_k \geqslant n/2}} \| Q^\delta \| \leqslant \sum_{\substack{k > 1 \\ i_1 + \dots + i_k < n/2}} \| Q^\delta \|$$

$$\times \varphi(n - i_1 - i_2 - \ldots - i_k) + 2 \sum_{\substack{k \geqslant 1 \\ i_1 + \dots + i_k \geqslant n/2}} \| Q^\delta \|$$

$$\leqslant \sum_{\substack{k \geqslant 1 \\ i_1 + \dots + i_k < n/2}} \| Q^\delta \| \varphi(n - i_1 - \ldots - i_k) + \sum_{\substack{k < \sqrt{n/2} \\ i_1 + \dots + i_k \geqslant n/2}} \| Q^\delta \|$$

$$+ \sum_{\substack{k \geqslant \sqrt{n/2} \\ i_1 + i_2 + \dots + i_k \geqslant n/2}} \| Q^\delta \| \leqslant \max_{n > m > n/2} \varphi(m) \sum_{k=0}^{\infty} (\| P_\nu - P_0 \| 2\varphi)^k$$

$$+ 2 \sum_{\substack{k < \sqrt{n/2} \\ i_1 + i_2 + \dots + i_k \geqslant n/2}} \varphi(i_1) \cdot \ldots \cdot \varphi(i_k) c^k \| P_\nu - P_0 \|^k$$

$$+ 2 \sum_{\substack{k \geqslant \sqrt{n/2} \\ i_1 + i_2 + \dots + i_k \geqslant n/2}} \varphi(i_1) \cdot \ldots \cdot \varphi(i_k) c^k \| P_\nu - P_0 \|^k \leqslant \max_{n > m > n/2} \varphi(m)$$

$$\times \sum_{k=0}^{\infty} (\| P_\nu - P_0 \| 2\varphi)^k + \sum_{\substack{k < \sqrt{n/2} \\ i_1 + i_2 + \dots + i_k > n/2}} \varphi(i_1) \cdot \ldots \cdot \varphi(i_k) c^k \| P_\nu - P_0 \|^k$$

$$+ \sum_{k \geqslant \sqrt{n/2}} (\| P_\nu - P_0 \| c\varphi)^k. \tag{4.10}$$

The first and third sums on the right side of (4.10) converge to zero as $n \to \infty$ because the series (4.8) is convergent and $\varphi(n)$ converges to zero as $n \to \infty$. Every term of the second sum contains a factor $\varphi(m)$ with $m > \sqrt{n/2}$. Therefore, the second sum is dominated by the sum

$$2 \max_{m > \sqrt{n/2}} \varphi(m) \sum_{k < \sqrt{n/2}} \| P_\nu - P_0 \| (c\varphi)^k ,$$

which converges to zero as $n \to \infty$. Thus (4.6) is proved.

It has remained to prove the analyticity of the vector $r_\nu = \lim_{n\to\infty} P_\nu^n x$ ($x \in M_1$). The absolute convergence of (4.6) implies the analyticity of r_ν in Q for $\|Q\|$ smaller than some q_0. The quantity Q is an analytic function of ν for $\nu \in D$. Therefore there is a ν_2 such that for $|\nu| < \nu_2$ we have $\|Q_\nu\| < q_0$, which implies the analyticity of r_ν in ν for $|\nu| < \nu_2$. The theorem is proved

As the theorem just proved shows, for the proof of the analyticity of the Markov chain family we have to construct sets M of uniform convergence for the operator P_0. An important class of Markov chains is the class of chains with exponential convergence (cf. [6], §27.3). Recall that a Markov chain has exponential convergence if there exist a set function $\tilde{P}(S)$ and positive constants a, $b > 0$ such that

$$| p^n(x, S) - \bar{P}(S) | \leqslant a \exp\{-bn\}$$

for sufficiently large n and any x and S. Any Markov chain with a finite set of states satisfies the condition of exponential convergence. In [5] and [6] conditions are given for the exponential convergence of countable Markov chains. For Markov chains with exponential convergence, as a set M of uniform convergence we may take the whole set of probability measures in the space $\mathfrak{X}(A, \Sigma)$. Therefore, for a family $\{L^\nu\}$ of Markov chains with exponential convergence the analyticity of the operator P_ν implies that of the stationary probabilities.

However, the Markov chains which are the most interesting in practice do not satisfy the condition of exponential convergence or even weaker conditions. For example, it is easy to show that for a birth and death process with a reflecting barrier at zero, i.e. a Markov proces with state set $\mathbf{Z}_+^1$ and transition probabilities $p_{i,i-1} = p$ and $p_{i,i+1} = 1 - p$, the set M (the whole set of probability measures) does not have the property of uniform convergence for any function $\varphi(n)$.

In the next section we give a method of constructing a set of uniform convergence for a class of Markov chains on which certain conditions are imposed in terms of "test" functions. These conditions are in particular satisfied for random walks in $\mathbf{Z}_+^n$.

§2. Analyticity conditions for a family of Markov chains in terms of "test" functions

Let us consider the same family $\{L^\nu\}$, $\nu \in D$, of Markov chains on the set $A = \{0, 1, \dots\}$ as in the preceding section. Assume that on A a family of real functions $f^\nu = \{f_i^\nu\}$ and a family of integral-valued positive functions $k^\nu = \{k_i^\nu\}$ ($i \in A$, $\nu \in D$) are given with the following properties:

1. $\inf_{i\in A,\, \nu\in D} f_i^\nu \geqslant 0$; $\sup_{i\in A,\, \nu\in D} k_i^\nu = b < \infty$.

2. For any fixed $b_1 > 0$ the series $\Sigma_{i=0}^{\infty} \exp\{-b_1 f_i^\nu\}$ converges uniformly in $\nu \in D$.

2'. Condition 2 is satisfied if there exist $c, \gamma > 0$ such that $f_i^\nu > ci^\gamma$ for any $\gamma \in D$.

3. There is a $d > 0$ such that $|f_i^\nu - f_j^\nu| > d$ implies that $p_{ij}(\nu) = 0$ ($\nu \in D$).

As before, $p_{ij}(t, \nu)$ is the t-step transition probability from the point i to the point j in L^ν.

THEOREM 4.2. *Let the operator P_ν defined by L^ν depend analytically on ν for $\nu \in D$. Assume that there exist n and $\delta > 0$ such that for any $i \in A$ and $j \in V_i = \{j: \sup_{\nu \in D} p_{ji}(\nu) > 0\}$ the inequality*

$$P_{ji}(n, 0) > \delta \tag{4.11}$$

holds and

$$\sum_{j=0}^{\infty} P_{ij}(k_i^\nu, \nu) f_j^\nu - f_i^\nu < -\varepsilon \tag{4.12}$$

for some $\varepsilon > 0$, all $\nu \in D$, and all i except some finite nonempty set B. Then the chains L^ν are ergodic for every $\nu \in D$, and the stationary probabilities $\pi_j(\nu)$ are analytic in ν for $|\nu|$ smaller than some ν_0 and for any $j \in A$.

PROOF. The ergodicity of the chains L^ν and the continuity of the stationary probabilities $\pi_j(\nu)$ ($j \in A$; $\nu \in D$) follow from Theorem 3.5. Let $\xi_0^\nu, \xi_1^\nu, \ldots$ be the sequence of the random variables corresponding to L^ν. We introduce

$$f_{ij}^n(\nu) = P\{\xi_1^\nu \neq j,\ \xi_2^\nu \neq j,\ \ldots,\ \xi_{n-1}^\nu \neq j,\ \xi_n^\nu = j/\xi_0^\nu = i\}.$$

$$_kP_{ij}(\nu) = P\{\xi_1^\nu \neq k,\ \xi_2^\nu \neq k,\ \ldots,\ \xi_{n-1}^\nu \neq k,\ \xi_n^\nu = j/\xi_0^\nu = i\},$$

$$_kP_{ij}^{*}(\nu) = \sum_{n=1}^{\infty} {}_kP_{ij}^n(\nu).$$

In Theorem 3.5 the estimate

$$f_{00}^n(\nu) < c_1 \exp\{-\delta_1 n\} \tag{4.13}$$

is proved for some $c_1, \delta_1 > 0$ and any $n \in A$ and $\nu \in D$. We can prove that

$$_0P_{0j} < c_1 \exp\{-\delta_1 n\}. \tag{4.14}$$

in an entirely analogous way. For convenience, we divide the proof of the theorem into a series of lemmas.

LEMMA 4.2. *The inequality*

$$\pi_j(\nu) < c_2 \exp\{-\delta_2 f_i^\nu\} \tag{4.15}$$

holds for some $c_2, \delta_2 > 0$ and any $i \in A$ and $\nu \in D$.

PROOF. For the irreducible aperiodic recurrent chain L^ν we have (cf. [1] or [8])

$$\lim_{m \to \infty} \frac{\sum_{n=0}^{m} p_{ij}(n, \nu)}{\sum_{n=0}^{m} P_{ii}(n, \nu)} = {}_iP_{ij}^{*}(\nu). \tag{4.16}$$

For an ergodic chain this becomes

$$\frac{\pi_j(\nu)}{\pi_i(\nu)} = {}_i p^{\bullet}_{ij}(\nu). \tag{4.17}$$

Consequently

$$\pi_j(\nu) < {}_0 p^{\bullet}_{0j}(\nu), \tag{4.18}$$

$${}_0 P^{\bullet}_{0j}(\nu) = \sum_{n=1}^{\infty} {}_0 P^{n}_{0j}(\nu) = \sum_{n=1}^{[f^{\nu}_{j}/d]} {}_0 p^{n}_{0j}(\nu) + \sum_{n=[f^{\nu}_{j}/d]}^{\infty} {}_0 p^{n}_{0j}(\nu). \tag{4.19}$$

The first sum on the right side of (4.19) is equal to zero. This follows from the hypothesis of Theorem 3. Using (4.14), we obtain that there exist constants c_2, $\delta_2 > 0$ such that

$$\sum_{n=[f^{\nu}_{j}/d]}^{\infty} {}_0 p^{n}_{0j}(\nu) < c_2 \exp\{-\delta_2 f^{\nu}_{j}\}. \tag{4.20}$$

Thus Lemma 4.2 is proved.

LEMMA 4.3. *There exist constants* c_3, $\delta_3 > 0$ *such that*

$$| p_{00}(n, \nu) - \pi_0(\nu) | < c_3 \exp\{-\delta_3 n\} \tag{4.21}$$

for any $n \in A$ *and* $\nu \in D$.

PROOF. We introduce the generating functions

$$F^{\nu}_{ij}(z) = \sum_{n=0}^{\infty} f^{n}_{ij}(\nu) z^n, \tag{4.22}$$

$$\mathbf{P}^{\nu}_{ij}(z) = \sum_{n=0}^{\infty} p_{ij}(n, \nu) z^n. \tag{4.23}$$

Using the relation

$$p_{ii}(n, \nu) = \sum_{s=1}^{n} f^{s}_{ii}(\nu) p_{ii}(n-s, \nu)$$

for the generating functions, we obtain

$$\mathbf{P}^{\nu}_{ii}(z) = \frac{1}{1 - F^{\nu}_{ii}(z)} \quad \text{or for } i = 0$$

$$\mathbf{P}^{\nu}_{00}(z) = \frac{1}{1 - F^{\nu}_{00}(z)}. \tag{4.24}$$

It follows from (4.13) that $F^{\nu}_{00}(z)$ is analytic in z for $|z| < 1 - \sigma$, for some $\sigma > 0$ and any $\nu \in D$. Besides, $|F^{\nu}_{ii}(z)| < 1$ for $|z| = 1$, $z \neq 1$, since the greatest common divisor k such that $f^{k}_{00}(\nu) \neq 0$ is 1. There exists a neighborhood U of $z = 1$ such that in it $F^{\nu}_{00}(z) = 1$ only for $z = 1$; therefore for some $\sigma_1 > 0$ the equation $F^{\nu}_{00}(z) - 1 = 0$ has no more roots for $|z| < 1 + \sigma_1$. Consequently $\mathbf{P}^{\nu}_{00}(z) = 1/(1 - F^{\nu}_{00}(z))$ is a meromorphic function for $|z| < 1 + \sigma_1$, and $z = 1$ is its only pole of first order (cf. [19], p. 237). The residue of $\mathbf{P}^\nu_{00}(z)$ at $z = 1$ is equal to

$$\operatorname*{res}_{z=1} \mathbf{P}^\nu_{00}(z) = \operatorname*{res}_{z=1} \frac{1}{1 - F^\nu_{00}(z)} = \lim_{z\to 1} \frac{z-1}{1 - F^\nu_{00}(z)} = -\lim_{z\to 1} \frac{1}{\dfrac{1 - F^\nu_{00}(z)}{1-z}}$$

$$= -\frac{1}{\left.\dfrac{dF^\nu_{00}(z)}{dz}\right|_{z=1}} = -\frac{1}{\sum\limits_{n=1}^{\infty} \dfrac{1}{f^n_{00}(\nu)\cdot n}} = -\pi^\nu_0.$$

Then $\tilde{\mathbf{P}}^\nu_{00}(z) = \mathbf{P}^\nu_{00}(z) - \pi^\nu_0/(1 - z)$ is holomorphic for $|z| < 1 + \varepsilon_2$. On the other hand, since

$$\tilde{\mathbf{P}}^\nu(z) = \sum_{n=0}^{\infty} (p_{00}(n, \nu) - \pi_0) z^n,$$

we conclude from this that there exist constants c_3, $\delta_3 > 0$ such that

$$| p_{00}(n, \nu) - \pi_0(\nu) | < c_3 \exp\{-\delta_3 n\}.$$

The lemma is proved.

LEMMA 4.4. *There exist constants* σ, c_4, $\delta_4 > 0$, *such that*

$$| p_{i0}(n, \nu) - \pi_0(\nu) | < c_4 \exp\{-\delta_4 n\} \tag{4.25}$$

for any $\nu \in D$, $i \in A$, *and* $n > \sigma f^\nu_i$.

PROOF. Using Lemma 1.2, we obtain analogously to (4.13) that there exist constants b_1, a_1, $\sigma_1 > 0$, such that

$$f^n_{i0}(\nu) < b_1 \exp\{-a_1 \cdot n\} \tag{4.26}$$

for any $\nu \in D$, $i \in A$ and $n > \sigma_1 \cdot f^\nu_i$. We have

$$p_{i0}(n, \nu) = \sum_{r=1}^{n} f^{n-r}_{i0}(\nu) p_{00}(r, \nu). \tag{4.27}$$

It follows from (4.27) that

$$| p_{i0}(n, \nu) - \pi_0(\nu) | \leqslant \sum_{r=1}^{n} | p_{00}(r, \nu) - \pi_0(\nu) | \cdot f^{n-r}_{i0}(\nu) + \pi_0 \sum_{r=n+1}^{\infty} f^r_{i0}(\nu). \tag{4.28}$$

Take $\sigma > \sigma_1$, and let $n > \sigma f^\nu_i$. We estimate the right side of (4.28).

1. Let $r < \varepsilon_1 n$, where $(1 - \varepsilon_1)\sigma > \sigma_1$. Then $(n - r) > (1 - \varepsilon_1)n > \sigma_1 f^\nu_i$. Therefore, using (4.26), we obtain

$$f^{n-r}_{i0}(\nu) < b_1 \exp\{-a_1(n-r)\} < b_1 \cdot \exp\{-a_1(1-\varepsilon_1)n\}.$$

2. Now let $r \geqslant \varepsilon_1 n$. Then Lemma 2 implies that

$$| p_{00}(r, \nu) - \pi_0(\nu) | < c_3 \exp\{-\delta_3 r\} < c_3 \exp\{-\delta_3 \varepsilon_1 n\}.$$

Combining both cases, we obtain

$$\sum_{r=1}^{n} | p_{00}(r, \nu) - \pi_0(\nu) | f_{i0}^{n-r}(\nu) < nb_2 \exp\{-a_2 n\} \tag{4.29}$$

for any $n > \sigma f_i^\nu$ and some $b_2, a_2 > 0$. It follows from (4.26) that

$$\pi_0 \sum_{r=n+1}^{\infty} f_{i0}^r(\nu) < b_3 \cdot \exp\{-a_3 n\}. \tag{4.30}$$

It follows from (4.29) and (4.30) that there exist constants c_4, $\delta_4 > 0$, such that (4.25) is satisfied for any $\nu \in D$, $i \in A$, and $n > \sigma f_i^\nu$. The lemma is proved.

LEMMA 4.5. *There exist constants* σ_1, c_5, $\delta_5 > 0$, *such that*

$$| p_{ij}(n, \nu) - \pi_j(\nu) | < c_5 \exp\{-\delta_5 n\} \tag{4.31}$$

for any $\nu \in D$, $i, j \in A$ *and* $n > \sigma_1 f_i^\nu$.

PROOF. We have

$$p_{ij}(n, \nu) = \sum_{r=1}^{n} p_{i0}(r, \nu)\, {}_0p_{0j}^{n-r}(\nu) + {}_0p_{ij}^n(\nu), \tag{4.32}$$

$$| p_{ij}(n, \nu) - \pi_j(\nu) | = \left| \sum_{r=1}^{n} (p_{i0}(r, \nu) - \pi_0(\nu))\, {}_0p_{0j}^{n-r}(\nu) \right.$$

$$\left. - \pi_0 \sum_{r=n-1}^{\infty} {}_0p_{0j}^r + {}_0p_{ij}^n(\nu) \right| \leqslant \sum_{r=1}^{n} | p_{i0}(r, \nu) - \pi_0(\nu) |\, {}_0p_{0j}^{n-r}(\nu)$$

$$+ \pi_0 \sum_{r=n+1}^{\infty} {}_0p_{0j}^r(\nu) + {}_0p_{ij}^n(\nu). \tag{4.33}$$

Using Lemma 4.4 and (4.14), as in Lemma 4.4, we can obtain (4.31) from (4.33).

LEMMA 4.6. *There exist constants* σ_2, c_6, $\delta_6 > 0$ *such that*

$$\sum_{j=0}^{\infty} | p_{ij}(n, \nu) - \pi_j(\nu) | < c_6 \exp\{-\delta_6 n\} \tag{4.34}$$

for any $\nu \in D$, $i, j \in A$ *and* $n > \sigma_2 f_i^\nu$.

PROOF. We have

$$\sum_{j=0}^{\infty} | p_{ij}(n, \nu) - \pi_j(\nu) | = \sum_{j: f_j^\nu > f_i^\nu + nd} | p_{ij}(n, \nu) - \pi_j(\nu) |$$

$$+ \sum_{j: f_j^\nu < f_i^\nu + nd} | p_{ij}(n, \nu) - \pi_j(\nu) |. \tag{4.35}$$

The boundedness of the jumps of the random walks implies that the first sum on the right side of (4.35) is equal to zero. It follows from Lemma 4.5 that each term of the second sum is less than $c_5 \exp\{-\delta_5 n\}$, whenever $n > \sigma_1 f_i^\nu$. Let M^ν be equal to the number of those j for which

$$f_j^\nu < f_i^\nu + nd. \tag{4.36}$$

Let j satisfy (4.36). Then

$$cj^{\gamma} < f_j^{\nu} < f_i^{\nu} + nd,$$

$$j < \left[\frac{1}{c}(f_i^{\nu} + nd)\right]^{1/\gamma} < \left[\frac{1}{c}\left(\frac{n}{\sigma_1} + nd\right)\right]^{1/\gamma} = n^{1/\gamma}\left[\frac{1}{c}\left(\frac{1}{\sigma_1} + d\right)\right]^{1/\gamma}.$$

Consequently,

$$M^{\nu} < n^{1/\gamma} b_1, \tag{4.37}$$

where

$$b_1 = \left[\frac{1}{c}\left(\frac{1}{\sigma_1} + d\right)\right]^{1/\gamma}.$$

Hence

$$\sum_{j: f_j^{\nu} < f_i^{\nu} + nd} | p_{ij}(n, \nu) - \pi_j(\nu) | < n^{1/\gamma} b_1 c_5 \exp\{-\delta_5 n\}.$$

Therefore, there exist constants σ_2, c_6, $\delta_6 > 0$ such that

$$\sum_{j=0}^{\infty} | p_{ij}(n, \nu) - \pi_j(\nu) | < c_6 \exp\{-\delta_6 n\}$$

whenever $n > \sigma_2 f_i^{\nu}$. The lemma is proved.

LEMMA 4.7. *There exist constants* δ_7, c_7, $\sigma_3 > 0$ *such that*

$$\sum_{n=1}^{\infty}\sum_{j=0}^{\infty} | p_{ij}(n, \nu) - \pi_j(\nu) | < \sigma_3 f_i^{\nu} + c_7 \exp\{-\delta_7 f_i^{\nu}\} \tag{4.38}$$

for any $\nu \in D$ *and* $i \in A$.

PROOF. We have

$$\sum_{n=1}^{\infty}\sum_{j=0}^{\infty} | p_{ij}(n, \nu) - \pi_j(\nu) | = \sum_{n: n < \sigma_2 f_i^{\nu}} \sum_{j=0}^{\infty} | p_{ij}(n, \nu) - \pi_j(\nu) |$$

$$+ \sum_{n: n > \sigma_2 f_i^{\nu}} \sum_{j=0}^{\infty} | p_{ij}(n, \nu) - \pi_j(\nu) |, \tag{4.39}$$

$$\sum_{j=0}^{\infty} | p_{ij}(n, \nu) - \pi_j(\nu) | < 2 \tag{4.40}$$

for any i, $n \in A$. Therefore, the first sum on the right side of (4.39) is less than $2\sigma_2 f_i^{\nu}$. We estimate the second sum on the right side of (4.39), using Lemma 4.6. Then we obtain

$$\sum_{n: n > \sigma_2 f_i^{\nu}} \sum_{j=0}^{\infty} | p_{ij}(n, \nu) - \pi_j(\nu) | < \sum_{n: n > \sigma_2 \cdot f_i^{\nu}} c_6 \exp\{-\delta_6 n\} < c_7 \exp\{-\delta_7 f_i^{\nu}\} \tag{4.41}$$

for some c_7, $\delta_7 > 0$. Combining the estimates thus obtained for the first and second sums on the right side of (4.39), we obtain the assertion of the lemma.

Now we begin the actual proof of Theorem 4.2. For some $\alpha > 1$ we introduce the set

$$M_\alpha = \left\{(x_0, x_1, \ldots, x_n \ldots): |x_i| \leqslant \alpha\pi_i(0); \sum_{i=1}^{\infty} x_i = 1\right\}. \quad (4.42)$$

We prove that M_α is a set of uniform convergence for the operator $P(0)$ corresponding to the Markov chain L^0. We prove that $P(0)M_\alpha \subset M_\alpha$. Let $\mathbf{x} = (x_0, x_1, \ldots, x_n) \in M_\alpha$. Set

$$P(0)\mathbf{x} = \mathbf{y} = (y_0, y_1, \ldots, y_n, \ldots),$$

$$|y_j| = \left|\sum_{i=0}^{\infty} p_{ij}(0)x_i\right| \leqslant \sum_{i=0}^{\infty} p_{ij}(0)\alpha\pi_i(0) = \alpha\pi_j(0).$$

Besides, $\Sigma_{j=0}^{\infty} y_j = 1$, $\mathbf{y} \in M_\alpha$. Set

$$\varphi_n = \sup_{\mathbf{x}\in M} \| P^n(0)\mathbf{x} - \pi(0) \|. \quad (4.43)$$

We have

$$\varphi_n = \sup_{\mathbf{x}\in M} \sum_{j=0}^{\infty} \left|\sum_{i=0}^{\infty} x_i p_{ij}(n, 0) - \pi_j(0)\right| = \sup_{\mathbf{x}\in M} \sum_{j=0}^{\infty} \left|\sum_{i=0}^{\infty} x_i(p_{ij}(n, 0) - \pi_j(0))\right|$$

$$\leqslant \sup_{\bar{\mathbf{x}}\in M} \sum_{i=0}^{\infty} |x_i| \sum_{j=0}^{\infty} |p_{ij}(n, 0) - \pi_j(0)| \leqslant \sum_{i=0}^{\infty} \alpha\pi_i(0) \sum_{j=0}^{\infty} |p_{ij}(n, 0) - \pi_j(0)|, \quad (4.44)$$

$$\sum_{n=1}^{\infty} \varphi_n < \alpha \sum_{n=1}^{\infty} \sum_{i=0}^{\infty} \pi_i(0) \sum_{j=0}^{\infty} |p_{ij}(n, 0) - \pi_j(0)|$$

$$= \alpha \sum_{i=0}^{\infty} \pi_i(0) \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} |p_{ij}(n, 0) - \pi_j(0)|. \quad (4.45)$$

To estimate the series (4.45) we use Lemmas 4.2 and 4.7. We have

$$\sum_{n=1}^{\infty} \varphi_n < \alpha \sum_{i=0}^{\infty} c_2 \exp\{-\delta_2 f_i^{\nu}\} (\sigma_3 f_i^{\nu_*} + c_7 \exp\{-\delta_7 f_i^{\nu}\}). \quad (4.46)$$

It follows from (4.46) that there exist constants a_1 and b_1 such that

$$\sum_{n=1}^{\infty} \varphi_n < a_1 \sum_{i=0}^{\infty} \exp\{-b_1 f_i^{\nu_*}\}. \quad (4.47)$$

The convergence of the series (4.47) follows from condition 2 of Theorem 4.2. By the same token, we have shown that M_α is a set of uniform convergence for the operator $P(0)$.

Take $\alpha_1 < \alpha_2$ and sets M_{α_1} and M_{α_2} of uniform convergence for the operator $P(0)$ such that $M_{\alpha_1} \subset M_{\alpha_2}$. Take $\mathbf{x} = (x_1, x_2, \ldots) \in M_{\alpha_1}$. We have $\Sigma_1^{\infty} x_i = 1$ and $x_i < \alpha_1\pi_i(0)$. Set $\mathbf{y} = (y_1, y_2, \ldots) = P(\nu)x$, where $y_i = \Sigma_{j=0}^{\infty} p_{ij}(\nu)x_j$. We have $\Sigma_0^{\infty} y_i = 1$, since P_ν is a Markov operator. The probability $p_{ji}(\nu)$ is different from zero only for $j \in V_i$ (the set V_i is defined in the hypothesis of the theorem). It follows from (4.11) that there is a constant $a > 0$ such that for all points $j \in V_i$ we have

$$\sum_{j\in V_i} \pi_j(0) < a\,\pi_i(0)$$

for any $i \in A$. Therefore

$$| y_i | \leqslant \sum_{j=0}^{\infty} p_{ji}(\nu)\, | x_j | \leqslant \sum_{j=0}^{\infty} p_{ji}(\nu)\, \alpha_1 \pi_j(0) < a\alpha_1\pi_i(0). \tag{4.48}$$

It follows from (4.48) that if $a\alpha_1 < \alpha_2$, then $y \in M_2$, and condition 2 of Theorem 4.1 is satisfied by the same token.

Let $\mathbf{x}^1, \mathbf{x}^2 \in M_{\alpha_1}$. We show then there exists ν_0 such that

$$\mathbf{z} = \mathbf{x}^1 + \frac{(P_\nu - P_0)\,\mathbf{x}^2}{\| P_\nu - P_0 \|} \in M_{\alpha_2}$$

for $|\nu| < \nu_0$, where $\mathbf{z} = (z_0, z_1, \dots)$. It is obvious that $\Sigma_0^\infty z_i = 1$. We show that $|z_i| < \alpha_2 \pi_i(0)$. We have

$$| z_i | \leqslant | x_i | + \frac{1}{\| P_\nu - P_0 \|} \left| \sum_{j=0}^{\infty} (P_{ji}(\nu) - p_{ji}(0))\, x_j \right|. \tag{4.49}$$

Since $P_{ji}(\nu)$ and $p_{ji}(0)$ are different from zero only for $j \in V_i$, and, as we have already shown, there exists an $\alpha > 0$ such that for all points $j \in V_i$ and any $i \in A$ we have $\Sigma_{j\in V_i} \pi_j(0) < a\pi_i(0)$, it follows that

$$| z_i | < \alpha_1\pi_i(0) + \frac{1}{\| P_\nu - P_0 \|} \max_{j\in V_i} | p_{ji}(\nu) - p_{ji}(0) |$$
$$\times \sum_{j\in V_i} \alpha_1 \pi_j(0) \leqslant \alpha_1\pi_i(0) + \alpha_1 a \pi_i(0) = \pi_i(0)\,\alpha_1(1 + a).$$

Setting $\alpha_2 > \alpha_1(1 + a)$, we obtain that $\mathbf{z} \in M_{\alpha_2}$. By the same token, the hypotheses of Theorem 4.1 are satisfied. This in turn implies the analyticity of the stationary probabilities. The theorem is proved.

§3. Analyticity of random walks in $\mathbf{Z}_+^n$

Consider a family $\{L^\nu\}$ of homogeneous irreducible aperiodic Markov chains with discrete time and set of states $\mathbf{Z}_+^n = \{(z_1, \dots, z_n)\colon z_i > 0, \text{ integral}\}$ ($\nu \in D$, where D is an open subset of the real line).

We shall assume that the homogeneity condition and the condition of boundedness of jumps introduced in §3 of Chapter III are satisfied. Besides, we assume that there exist n, $\delta > 0$ such that for any $\nu \in D$, $\alpha \in \mathbf{Z}_+^n$ and $\beta \in V_\alpha$, where $V_\alpha = \{\beta\colon \sup_{\nu\in D} p_{\beta\alpha}(\nu) > 0\}$, we have

$$p_{\beta\alpha}(n, \nu) > \delta.$$

As in §3 of Chapter III, we introduce the family $\{V^\nu\}$ of vector fields.

THEOREM 4.3. *Assume that the operator* P_ν *defined by the chain* L^ν *depends on* ν *analytically for* $\nu \in D$. *If there exists a set* $U \subset D$ *such that the family* $\{V^\nu\}$, $\nu \in D$, *of vector fields satisfies condition* $\tilde{\mathrm{B}}$ (*see* §3 *of Chapter* III), *then the chains* L^ν *are ergodic for all* $\nu \in U$, *and the stationary probabilities* $\pi_\alpha(\nu)$ *are analytic in* ν *for any* $\alpha \in \mathbf{Z}_+^n$ *and* $\nu \in U$.

PROOF. As in §3 of Chapter III, condition $\tilde{\mathrm{B}}$ implies the existence of a function $f^\nu(\alpha)$, $\nu \in D$, which satisfies (4.12). Besides, the families of functions $\{k_i^\nu\}$ and $\{f_i^\nu\}$ satisfy conditions 1, 2, and 3 imposed on them in Theorem 4.2. Inequality (4.11) is satisfied by assumption. Hence, all hypotheses of Theorem 4.2 are satisfied, which leads to the analyticity of $\pi_\alpha(\nu)$ for $\alpha \in \mathbf{Z}_+^n$ and $\nu \in U \subset D$.

For random walk families $\{L^\nu\}$ in $\mathbf{Z}_+^n$, where $n \leqslant 3$, the analyticity conditions for the stationary probabilities can be formulated explicitly, since we have succeeded in constructing a function $f(\alpha)$ satisfying condition $\tilde{\mathrm{B}}$ for the family $\{L^\nu\}$. Let us formulate this theorem for walks in $\mathbf{Z}_+^3$.

THEOREM 4.4. *Assume that the Markov chain* L^{ν_0}, *where* $\nu_0 \in D$, *satisfies the hypotheses of Theorem* 2.4 *guaranteeing the ergodicity of* L^ν. *Moreover, assume that the operator* P_ν *depends analytically on* ν *for* $\nu \in D$. *Then there is a neighborhood* U *of* ν_0 ($U \subset D$) *such that the chains* L^ν *are ergodic for all* $\nu \in U$, *and the stationary probabilities* $\pi_\alpha(\nu)$ *are analytic in* ν *for any* $\alpha \in \mathbf{Z}_+^3$ *and* $\nu \in U$.

The proof of this theorem is completely analogous to that of Theorem 3.7, and we shall not include it here.

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BIBLIOGRAPHY

1. Kai Lai Chung, *Markov chains with stationary transition probabilities*, Springer-Verlag, 1960.
2. Frank Spitzer, *Principles of random walk*, Van Nostrand, Princeton, N. J., 1964.
3. R. Z. Has'minskiĭ, *Stability of differential equations under random perturbations of their parameters*, "Nauka", Moscow, 1969. (Russian)
4. V. A. Malyšev, *Random walks. Wiener-Hopf equations in a quarter-plane. Galois automorphisms*, Izdat. Moskov. Gos. Univ., Moscow, 1970. (Russian)
5. J. L. Doob, *Stochastic processes*, Wiley, New York, and Chapman & Hall, London, 1953.
6. Michel Loève, *Probability theory*, 2nd ed., Van Nostrand, Princeton, N. J., 1960.
7. V. A. Malyšev, *Wiener-Hopf equations and their application in probability theory*, Itogi Nauki: Teor. Verojatnost., Mat. Statist., Teoret. Kibernet., vol. 13, VINITI, Moscow, 1975, pp. 5–39; English transl. in J. Soviet Math. 7 (1977), no. 2.
8. Samuel Karlin, *A first course in stochastic processes*, Academic Press, 1966.
9. M. V. Men'šikov, *Ergodicity and transience conditions for random walks in the positive octant of space*, Dokl. Akad. Nauk SSSR **217** (1974), 755–758; English transl. in Soviet Math. Dokl. **15** (1974).
10. V. A. Malyšev and M. V. Men'šikov, *Ergodicity in three-dimensional Markov problems of queueing theory*, Queueing Theory (Proc. Third All-Union School-Conf., Puščino-on-Oke, 1974), Vol. 1, Izdat. Moskov. Gos. Univ., Moscow, 1966, pp. 128–134. (Russian)
11. ———, *Continuous stationary probability conditions for Markov chains*, Third Soviet-Japanese Sympos. Probability Theory (Tashkent, 1975), Abstracts of Reports, Part I, "Fan", Tashkent, 1975, pp. 34–36. (Russian)
12. V. A. Malyšev, *Classification of two-dimensional positive random walks and almost linear semimartingales*, Dokl. Akad. Nauk SSSR **202** (1972), 526–528; English transl. in Soviet Math. Dokl. **13** (1972).

13. E. B. Dynkin, *Boundary theory of Markov processes (the discrete case)*, Uspehi Mat. Nauk **24** (1969), no. 2 (146), 3–42; English transl. in Russian Math. Surveys **24** (1969).

14. B. V. Gnedenko and I. N. Kovalenko, *Introduction to queueing theory*, "Nauka", Moscow, 1966; English transl., Israel Program Sci. Transls., Jerusalem, and Davey, Hartford, Conn., 1968.

15. V. A. Malyšev, *Nonstandard Markov systems in queueing theory*, Queueing Theory, Ju. K. Beljaev, editor (Proc. Second All-Union School-Conf., Diližan, 1971), Izdat. Moskov. Gos. Univ., Moscow, 1972, pp. 91–95. (Russian)

16. V. M. Zolotarev, *Quantitative estimates in problems on the continuity of queuing systems*, Teor. Verojatnost. i Primenen. **20** (1975), 215–218; English transl. in Theor. Probability Appl. **20** (1975).

17. V. V. Kalašnikov, *Uniform closeness of components of a Markov process with variation of its parameters*, Izv. Akad. Nauk SSSR Tehn. Kibernet. **1974**, no. 4, 43–53; English transl. in Engrg. Cybernetics **12** (1974).

18. ______, *The property of γ-reflexivity for Markov sequences*, Dokl. Akad. Nauk SSSR **213** (1973), 1243–1246; English transl. in Soviet Math. Dokl. **14** (1973).

19. B. V. Šabat, *Introduction to complex analysis*, "Nauka", Moscow, 1969. (Russian)

20. Z. Navickas, *Criteria for the ergodicity of homogeneous Markov chains in a special phase space*. I, II, Litovsk. Mat. Sb. **12** (1972), no. 3, 113–129; 131–146. (Russian)

21. ______, *Criteria for the ergodicity of homogeneous Markov chains in a special phase space*. III, Litovsk. Mat. Sb. **12** (1972), no. 4, 153–174. (Russian)

22. Ju. V. Prohorov, *Convergence of random processes and limit theorems of probability theory*, Teor. Verojatnost. i Primenen. **1** (1956), 177–238; English transl. in Theor. Probability Appl. **1** (1956).

23. F. G. Foster, *On the stochastic matrices associated with certain queuing processes*, Ann. Math. Statist. **24** (1953), 355–360.

24. A. Shimizu, *On the stability of a discrete time Markov chain*, Report given at the Second Japan-USSR Symposium on Probability Theory (Kyoto, August 1972).

25. William F. Stout, *Maximal inequalities and the law of the iterated logarithm*, Ann. Probability **1** (1973), 322–328.

26. ______, *A martingale analogue of Kolmogorov's law of the iterated logarithm*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **15** (1970), 279–290.

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