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Hydrodynamics of the Weakly Perturbed Voter Model

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Abstract. We study the Euler and diffusion limits for a weakly perturbed voter model. We present a new method for deriving the hydrodynamics and we obtain the hydrodynamic equations for a large class of perturbations.

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Introduction

We derive the hydrodynamic equations for a weakly perturbed voter model. Thus we add one more model of a stochastic particle system to the now existing list of models (see references in [1, 2, 19]), for which the hydrodynamics have so far been established. There are however more serious reasons for presenting this paper. They are as follows.

(i) We present a new method for deriving hydrodynamic equations. It uses perturbation expansions as in [3] (see also [2]) but in a completely different way. In particular, we do not need any theorems on the existence of solutions to the limiting partial differential (hydrodynamic) equations. In fact, the existence of such solutions is proved by our method. Our method gives a complete control of each term of the expansion. For other methods see e.g. [4,8] and the bibliography in [2].

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(ii) The conservation law for the voter model is weaker than, for example, for the simple exclusion process. Nevertheless, the hydrodynamic equations hold in the linear case, as has been shown by Presutti and Spohn [17]. Our paper thus generalises [17] to the weakly nonlinear case.

(iii) Some results exist now ([11, 16]) on the large time behaviour of weakly nonlinear voter models (with perturbation parameter ε fixed, $t \to \infty$). This allows us to go further with respect to hydrodynamical time and to compare stationary solutions of hydrodynamic equations and invariant measures for the initial particle system (see [12, 13]).

The paper is organised as follows. We consider the voter model in discrete time. This allows us to use earlier techniques for marginally closed processes [7, 15]. Sections 1, 2 and 3 discuss the case of positive nonlinear perturbations of the voter model. Section 1 considers Euler scaling. We describe the dual process and study some properties of its trajectories (diagrams). We derive the hydrodynamic equation, which is a first order quasi-linear partial differential equation. Section 2 considers diffusion scaling. The hydrodynamic equation is a second order parabolic equation (the heat equation with nonlinear source). The law of large numbers for the generalised random fields generated by the nonlinear perturbed voter model is given in Section 3. Section 4 discusses the case of general (non-positive) perturbations. Additional difficulties arise in this case because of the absence of a probabilistic dual process. Generalisations of all results on positive perturbations to the case of non-positive perturbations are presented.

As a conclusion let us note that our method is also applicable to the continuous time model and that the main ideas of the proof are the same.

This paper is a revised version of [14].

Discrete time voter model

A discrete time voter model is a Markov process ξ_t with local interactions and with values in $\{0,1\}^{\mathbf{Z}^\nu}$, such that

1) it is conditionally independent [7]: for any $x_1, \dots, x_n \in \mathbf{Z}^\nu$

$$\mathsf{P}\{\xi_t(x_1) = 1, \dots, \xi_t(x_n) = 1 | \xi_{t-1}(z), z \in \mathbf{Z}^\nu\} =$$

$$= \mathsf{P}\{\xi_t(x_1) = 1 | \xi_{t-1}(z), z \in \mathbf{Z}^\nu\} \cdots \mathsf{P}\{\xi_t(x_n) = 1 | \xi_{t-1}(z), z \in \mathbf{Z}^\nu\}$$

(i.e. the components $\xi_t(x)$, $x \in \mathbf{Z}^\nu$, of ξ_t are conditionally independent given ξ_{t-1});

2) it has the following conditional probabilities:

$$\mathsf{P}\{\xi_t(x) = 1 | \xi_{t-1}(z), z \in \mathbf{Z}^\nu\} = \sum_{y \in Q} a_y \xi_{t-1}(x+y),$$

where Q is a fixed finite subset of $\mathbf{Z}^\nu$, and

$$a_y \geq 0, \qquad \sum_{y \in Q} a_y = 1.$$

A complete description of a continuous time variant of this model can be found in [10].

It can be easily seen from the above definition that for any x the mean value of $\xi_t(x)$ is conserved under the dynamics of the system. A fundamental result concerning the ergodic behaviour of ξ_t is the following.

Let $\mu_t^{(\alpha)}$ denote the law of the process ξ_t with initial translation invariant Bernoulli product measure with mean α, $0 \leq \alpha \leq 1$. Then $\mu_t^{(\alpha)}$ converges weakly to a measure μ_α, as $t \to \infty$.

The limiting measure μ_α is not a product measure. Indeed,

$$\mu_\alpha = (1-\alpha)\delta_0 + \alpha\delta_1$$

for $\nu \leq 2$, where δ_0 and δ_1 are the measures concentrated on a single configuration: $\xi(x) \equiv 0$ and $\xi(x) \equiv 1$, respectively. For $\nu \geq 3$, μ_α is some non-trivial translation invariant measure with mean α and slowly decaying correlations. For example, the asymptotics of the two point covariance function is given by

$$\begin{aligned} \mathrm{Cov}(\xi(x), \xi(y)) &\equiv \langle \xi(x)\xi(y) \rangle_{\mu_\alpha} - \langle \xi(x) \rangle_{\mu_\alpha} \langle \xi(y) \rangle_{\mu_\alpha} \\ &\sim C(\alpha)|x-y|^{-\nu+2}, \end{aligned}$$

for $|x-y|$ large. The n-point correlation function can be expressed in terms of the trajectories of n-particle coalescing random walks on a lattice [7, 15].

The duality relation between the voter model and a system of coalescing random walks has been effectively exploited by many authors, see [10] and references therein, [7, 15]. This duality explains the dependence of the ergodic behaviour of the voter model on the dimension. In this paper we extend this duality technique to a class of perturbed voter models through a system of branching and coalescing random walks with possible death of particles.

In [17] Presutti and Spohn presented a hydrodynamical description of the voter model. They considered a continuous time variant of the model with equal a_y and a family μ_ε of initial measures with slowly (of order ε) varying mean value, $\varepsilon \to 0$. They proved that the generalised field corresponding to the process under the diffusion scaling converges weakly to a deterministic field that is the solution to a diffusion equation.

1. Hydrodynamical limit for the voter model with a positive perturbation

1.1. Euler approximation

Definition. A discrete time Markov process ξ_t with values in $\{0,1\}^{\mathbf{Z}^\nu}$ is called a voter model with positive peturbation if it is conditionally independent with conditional probabilities of the form:

$$\begin{aligned} \mathsf{P}\left\{\xi_t(x)=1|\xi_{t-1}(z), z\in \mathbf{Z}^\nu\right\} &= (1-C\varepsilon)\sum_{y\in Q} a_y \xi_{t-1}(x+y) \\ &\quad + \varepsilon \sum_{A\in\mathcal{A}} c_A \prod_{y\in A} \xi_{t-1}(x+y) + \varepsilon\beta, \end{aligned} \tag{1.1}$$

for Q a fixed finite subset of $\mathbf{Z}^\nu$, $\varepsilon > 0$ a small parameter, a_y satisfying

$$a_y \geq 0, \qquad \sum_{y\in Q} a_y = 1, \tag{1.2}$$

$\mathcal{A}$ a finite collection of finite nonempty subsets of $\mathbf{Z}^\nu$, and

$$C = \sum_{A\in\mathcal{A}} c_A + \beta.$$

Further consider a discrete time homogeneous random walk on $\mathbf{Z}^\nu$ with jump probabilities

$$p(y) = \begin{cases} a_y, & \text{if } y \in Q, \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

For simplicity we assume that this random walk is *irreducible* and *aperiodic*.

Let $\vec{a} = (a^1, \ldots, a^\nu) \in \mathbf{R}^\nu$ be the vector of mean jumps

$$\vec{a} = \sum_y y\, p(y). \tag{1.4}$$

Let $A = \{z_1, \ldots, z_k\}$ be a finite subset of $\mathbf{Z}^\nu$ and let $A = B_1 \cup \ldots \cup B_j$ be a partition of A into non-empty non-intersecting subsets (we call them blocks)

$$B_1, \ldots, B_j, \quad (B_l \cap B_m = \emptyset,\ l \neq m;\ |B_l| \geq 1 \text{ for all } l).$$

Consider k coalescing random walks $z_i(t), i = 1, \ldots, k$, with jump probabilities (1.3), starting at z_i, $z_i = z_i(0)$. Each particle moves independently of the others until it meets another particle. Whenever two particles meet, they coalesce into one particle. We denote by $D_{B_1,\ldots,B_j}$ the probability that for each $l \in \{1, \ldots, j\}$ all random walkers $z_i(t)$, $z_i \in B_l$, from the same block l coalesce into one particle, whilst random walkers from different blocks do not coalesce.

For any one-point set A, $|A| = 1$, we put $D_A = 1$ by definition.

Remark. $D_{B_1,\ldots,B_j} \equiv 0$ for all $j > 1$ in dimensions $\nu = 1, 2$. This follows from the fact that the one- and two-dimensional symmetric random walks are recurrent.

Let

$$N = \max\{|A| : A \in \mathcal{A}\},$$

and for all $j \leq N$

$$d_j = \sum_{A \in \mathcal{A}, |A| \geq j} c_A \sum_{B_1 \cup \ldots \cup B_j = A} D_{B_1,\ldots,B_j}. \tag{1.5}$$

The inner sum in the last expression is taken over all different partitions of the set A into j non-empty non-intersecting subsets

$$B_1, \ldots, B_j \quad (B_i \cap B_k = \emptyset,\ i \neq k).$$

For any $r = (r^1, \ldots, r^\nu) \in \mathbf{R}^\nu$ denote by $[r]$ the vector $([r^1], \ldots, [r^\nu]) \in \mathbf{Z}^\nu$. Denote by ∇_r the operator $\left(\dfrac{\partial}{\partial r^1}, \ldots, \dfrac{\partial}{\partial r^\nu}\right)$.

Theorem 1.1. *Let $\langle\cdot\rangle$ denote the expectation operator of the Markov process $\xi_t(\cdot)$ generated by* (1.1), (1.2), *having an initial product measure with a slowly varying parameter, i.e.*

$$\langle \xi_0(x) \rangle = \rho_0(\varepsilon x), \tag{1.6}$$

where $\rho_0 : \mathbf{R}^\nu \to [0, 1]$, $\rho_0 \in C^1(\mathbf{R}^\nu)$.

Then for any ν, for any $r \in \mathbf{R}^\nu$ and any $\tau \in \mathbf{R}_+$ the limit

$$\lim_{\varepsilon \to 0} \langle \xi_{[\tau/\varepsilon]}\big(\big[\frac{\tau}{\varepsilon}\big]\big) \rangle = \rho(\tau, r) \tag{1.7}$$

exists. The function $\rho(r, \tau)$ belongs to $C^1(\mathbf{R}^\nu \times \mathbf{R}_+)$ and satisfies the equation

$$\frac{\partial \rho}{\partial \tau} = (\vec{a}, \nabla_r \rho) - C\rho + \sum_{j=1}^{N} d_j \rho^j + \beta \tag{1.8}$$

with initial condition

$$\rho(0, r) = \rho_0(r). \tag{1.9}$$

Example. Consider the special case of

$$\begin{aligned} \mathsf{P}\{\xi_t(x) = 1 | \xi_{t-1}(z), z \in \mathbf{Z}^\nu\} &= (1 - (\alpha + \beta)\varepsilon) \sum_{y \in Q} a_y \xi_{t-1}(x + y) \\ &\quad + \varepsilon\, \alpha \xi_{t-1}(x + y')\xi_{t-1}(x + y'') + \varepsilon\beta, \end{aligned} \tag{1.10}$$

for fixed vectors y' and y'' in $\mathbf{Z}^\nu$. Then the hydrodynamical equation is given by

$$\begin{aligned}\frac{\partial\rho(\tau,r)}{\partial\tau} &= (\vec{a},\nabla_r)\rho(\tau,r) \\ &\quad + (\alpha D-(\alpha+\beta))\rho(\tau,r)+\alpha(1-D)\rho^2(\tau,r)+\beta,\end{aligned} \tag{1.11}$$

with $D = D_{\{y',y''\}}$ the probability that the two independent random walks in $\mathbf{Z}^\nu$ starting at sites y' and y'' respectively and with jump probabilities given by (1.3), will ever meet. In the case of $\alpha = 1,\ \ \beta = 0$ we get

$$\frac{\partial\rho(\tau,r)}{\partial\tau} = \sum_{i=1}^{\nu} a^i \frac{\partial\rho(\tau,r)}{\partial r^i} - (1-D)(\rho(\tau,r)-\rho^2(\tau,r)). \tag{1.12}$$

The remainder of this section will be devoted to the proof of Theorem 1.1. We use a graphical representation and the dual process. This dual process is a branching coalescing random walk with some probability (in general non-zero) of death. The main point is that the mean number of particles during time ε^{-1} is uniformly bounded in ε.

Proof of Theorem 1.1. Without loss of generality we prove the Theorem for the case considered in the Example. The generalisation is evident.

1.2. The diagrams

A diagram $G = G(t,x)$ is a graph with vertices in $\mathbf{Z}_t = \{0,1,\ldots,t\}\times\mathbf{Z}^\nu$. A k-slice of $\mathbf{Z}_t$ will be the set $\{k\}\times\mathbf{Z}^\nu$, $k = 0,1,\ldots,t$. An edge of the diagram connects two vertices on sequential slices of $\mathbf{Z}_t$. We construct the diagrams sequentially slice by slice, starting with the t-slice.

Algorithm for constructing diagrams

1. Each diagram has a unique vertex on the t-slice, namely (t,x).
2. There are three possibilities for creating verteces on the $(t-1)$-slice and edges connecting them with the t-slice:
   (i) One vertex on the $(t-1)$-slice. This is a vertex $(t-1,x+y)$ for some $y\in Q$. In this case there is an edge connecting this vertex with (t,x); it is called *a single edge*.
   (ii) Two vertices on the $(t-1)$-slice, namely $(t-1,x+y')$ and $(t-1,x+y'')$. Then there are two edges connecting these two vertices with (t,x). In this case we will call this *a branching* at x at time t.
   (iii) No vertices on the $(t-1)$-slice. Then we will call (t,x) *a final vertex*.

3. Suppose that the k-slice of the diagram has been constructed. For each vertex on the k-slice we construct one, two or zero vertices (and correspondingly one, two or zero edges) on the $(k-1)$-slice using (i), (ii), or (iii). We do not take into account the multiplicity of a vertex, i.e. if some vertex has been constructed two or more times, it is considered to be a single vertex.

4. The procedure terminates on the l-slice, if all vertices on the $l+1$-slice are final. Otherwise the procedure terminates on the 0-slice. We call all vertices belonging to the 0-slice *end-point vertices*.

We define the contribution $J(G)$ of a diagram G as follows:

$$J(G) = (1 - \varepsilon(\alpha+\beta))^s (\varepsilon\alpha)^b (\varepsilon\beta)^f \prod_y a_y \, \langle \xi_0(z_1)\rangle \ldots \langle \xi_0(z_m)\rangle, \tag{1.13}$$

where $s = s(G)$ is the total number of single (i.e. not being a part of any branching) edges in the diagram G, $b = b(G)$ is the number of branchings in G, $f = f(G)$ is the number of final vertices in G, $\prod_y$ is the product over all edges $\{(t', x'), (t'-1, x'+y)\}$ of G without branching in x' and $z_1, \ldots, z_m$ are the end-point vertices of G on the 0-slice.

Lemma 1.1.

$$\langle \xi_t(x)\rangle = \sum_G J(G) \tag{1.14}$$

where the summation is over all diagrams constructed with the above algorithm and where $J(G)$ are defined in (1.13)*.*

Proof. Taking the expectation in (1.10) and iterating this t times, we get (1.14). □

1.3. The dual process

It is convenient to view these diagrams as the trajectories (in reverse time $\bar{t} = [\tau/\varepsilon] - t$) of a coalescing branching random walk $\eta_{\bar{t}}$ on $\mathbf{Z}^\nu$. The random walk $\eta_{\bar{t}}$, starting at $x \in \mathbf{Z}^\nu$, jumps to the point $x + y$, $y \in Q$, with probability $(1 - \varepsilon(\alpha+\beta))a_y$; with probability $\varepsilon\alpha$ it produces a new particle (all particles are identical) that is put at point $x + y''$, and the particle itself jumps to $x + y'$; the particle can also die, and this occurs with probability $\varepsilon\beta$. If the number of particles is greater than 1 they behave independently of each other. Note that two particles coalesce when they jump to the same point x' at the same moment of time, i.e. they become a single particle. This occurs because of the property $(\xi_t(x'))^2 = \xi_t(x')$.

We need another stochastic process $\zeta_{\bar{t}}$ majorising $\eta_{\bar{t}}$. It is a discrete time branching process with one type of particles. We assume that each of the particles in $\zeta_{\bar{t}}$ produces at the next time instant and independently of the other particles *zero*, *one* or *two* descendants with probabilities:

$$p_0 = \varepsilon\beta, \quad p_1 = 1 - \varepsilon(\alpha + \beta), \quad p_2 = \varepsilon\alpha. \tag{1.15}$$

Lemma 1.2. *The mean number of branchings of the process $\eta_{[\tau/\varepsilon]}$ is uniformly bounded in ε .*

Proof. It is easy to see that the mean number of branchings $\mathsf{E}b(\eta_{\bar{t}})$ of the process $\eta_{\bar{t}}$ is not larger than the mean number of branchings $\mathsf{E}b(\zeta_{\bar{t}})$ of the pure branching process $\zeta_{\bar{t}}$, i.e.

$$\mathsf{E}b(\eta_{\bar{t}}) \leq \mathsf{E}b(\zeta_{\bar{t}}). \tag{1.16}$$

The mean number of particles for $\zeta_{\bar{t}}$ at time 1 is equal to

$$m_\varepsilon = 1 - \varepsilon(\alpha + \beta) + 2\varepsilon\alpha = 1 + \varepsilon(\alpha - \beta). \tag{1.17}$$

The mean number of particles at time t for $\zeta_{\bar{t}}$ is equal to $m_{\varepsilon,t} = m_\varepsilon^t$ (cf. [6]). Hence the mean number of branchings for $\eta_{\bar{t}}$ at time $\bar{t} = [\tau/\varepsilon]$ is uniformly bounded in ε, even if the process $\zeta_{\bar{t}}$ shows a super-critical behaviour. This is because

$$m_{\varepsilon,\tau} = (1 + \varepsilon(\alpha - \beta))^{[\tau/\varepsilon]} \leq e^{\tau(\alpha-\beta)}.$$

□

Corollary 1.1. *The probability that the number of branchings of $\eta_{[\tau/\varepsilon]}$ is bigger than N tends to 0 as $N \to \infty$, uniformly in ε.*

Proof. This follows from Lemma 1.2 by Chebyshev's inequality

$$P(b(\eta_{[\tau/\varepsilon]}) > N) \leq \frac{m_{\varepsilon,\tau}}{N}. \tag{1.18}$$

□

1.4. The limit

Denote by $\mathcal{G}_n = \mathcal{G}_n([\tau/\varepsilon], [r/\varepsilon])$ the class of diagrams with exactly n branchings constructed for $\langle \xi_{[\tau/\varepsilon]}([r/\varepsilon]) \rangle$. Let us represent the sum (1.14) as a series in the number of branchings:

$$\langle \xi_{[\tau/\varepsilon]}([\frac{r}{\varepsilon}]) \rangle = \sum_{n=0}^{\infty} \sum_{G \in \mathcal{G}_n} J(G). \tag{1.19}$$

Lemma 1.2 implies that this series converges uniformly in ε. We want to take a termwise limit in (1.19) for $\varepsilon \to 0$. Let us fix N large enough, such that $\sum_{n>N} \sum_{G\in\mathcal{G}_n} J(G)$ is small. The main difficulty is to calculate the limits of the diagrams having vertices on the 0-slice, since these diagrams "remember" initial conditions. This is the reason for restricting to the case $\alpha = 1,\ \beta = 0$ in the present and next subsections.

To formulate the main Lemma we need some definitions. Let us refer to branchings and coalescings as *events*. We define a *path* as a subgraph of G which has vertices on consequtive slices; one vertex on each slice:

$$\{(t_1, x_1), (t_1 - 1, x_2), \ldots, (t_1 - k + 1, x_k)\},$$

connected sequentially by edges.

For any given diagram G with n branchings, k coalescings and m vertices on the 0-slice we construct an abstract graph Γ with $n + k + m + 1$ vertices in the following way. We number all vertices of G, in which a branching or coalescing occurs, in their order of appearance in dual time from 2 to $n+k+1$. We number the endpoints of G from $n + k + 2$ to $n + k + m + 1$. Vertex 1 of Γ corresponds to the initial vertex $([\tau/\varepsilon], [r/\varepsilon])$ of G; the other vertices of Γ correspond to the numbered vertices of G. We construct an edge connecting vertices i and j of Γ, if there exists a path in G connecting the corresponding vertices of G and not containing any other numbered vertex of G. If two such paths exist, we construct two edges between i and j. (Note, that no more than two such paths exist, since a branching gives birth to only one new particle).

Let $\mathcal{T}_n$ be the set of graphs Γ generated by the diagrams in $\mathcal{G}_n$.

Lemma 1.3. *Consider two functions $x' = x'(\varepsilon) \in \mathbf{Z}^\nu$ and $t' = t'(\varepsilon) \in \mathbf{Z}$, such that*

$$\begin{aligned} \varepsilon x'(\varepsilon) &\to 0, \quad \varepsilon \to 0, \\ \varepsilon t'(\varepsilon) &\to 0, \quad \varepsilon \to 0. \end{aligned} \tag{1.20}$$

Choose initial vertices $([\tau/\varepsilon] + t', [r/\varepsilon] + x')$ in the diagrams. Then for any $n \le N$ and any $\Gamma \in \mathcal{T}_n$ the following limit exists:

$$\lim_{\varepsilon\to 0} \sum_{G\,:\,\Gamma} J(G),$$

where the summation is over all diagrams $G \in \mathcal{G}_n$ corresponding to a given Γ by the above procedure. This limit does not depend on x' and t'.

For $\alpha = 1,\ \beta = 0$ the first statement (1.7) of Theorem 1.1 follows from (1.19), Lemma 1.2, Corollary 1.1 and Lemma 1.3. To prove (1.8) we need an explicit expression for the limit in Lemma 1.3, which we will derive in the course of the proof.

Proof of Lemma 1.3. We use induction to the number of branchings. For the induction step we will not only need to prove the existence of the limit, but also that the limit is invariant under a slight perturbation $([\tau/\varepsilon]+t', [r/\varepsilon]+x')$ of the initial point $([\tau/\varepsilon], [r/\varepsilon])$.

The basis of the induction

1. No branchings

For $n = 0$, $\mathcal{T}_0$ consists of a single graph with two vertices and an edge connecting these. We will call $G \in \mathcal{G}_0$ a *path*.

Lemma 1.4. *Consider the functions $x'(\varepsilon)$ and $t'(\varepsilon)$ satisfying* (1.20). *Then*

$$\lim_{\varepsilon\to 0} \sum_{G\in\mathcal{G}_0([\tau/\varepsilon]+t',[r/\varepsilon]+x')} J(G) = \lim_{\varepsilon\to 0} \sum_{G\in\mathcal{G}_0([\tau/\varepsilon],[r/\varepsilon])} J(G) = e^{-\tau}\rho_0(r+\vec{a}\tau). \tag{1.21}$$

Proof.

$$\sum_{G\in\mathcal{G}_0([\tau/\varepsilon],[r/\varepsilon])} J(G) = (1-\varepsilon)^{[\tau/\varepsilon]} \sum_{z\in\mathbf{Z}^\nu} P_{[\tau/\varepsilon]}\big(\big[\frac{r}{\varepsilon}\big] \to z\big)\rho_0(\varepsilon z) \tag{1.22}$$

with $P_t(x \to z)$ the probability that the random walk with jumps given by (1.3) and initial point x at time 0, is in z at time t. An easy consequence of the law of large numbers and the continuity of ρ_0 is:

$$\lim_{\varepsilon\to 0} \sum_{z\in\mathbf{Z}^\nu} P_{[\tau/\varepsilon]}\big(\big[\frac{r}{\varepsilon}\big] \to z\big)\,\rho_0(\varepsilon z) = \rho_0(r+a\tau).$$

This remains true, if we take $[r/\varepsilon]+x'$ and $[\tau/\varepsilon]+t'$ instead of $[r/\varepsilon]$ and $[\tau/\varepsilon]$. This proves Lemma 1.4. □

2. One branching

Next we consider diagrams with exactly one branching. There are two such types of diagrams: branching with coalescing, which we will call *a loop* (Figure 1) and branching without coalescing, which we will call *an actual branching* (Figure 2). These diagrams play a basic role in our study.

Denote by f_t the probability that two independent random walks $w_t^{(1)}$ and $w_t^{(2)}$ on $\mathbf{Z}^\nu$ with jump probabilities (1.3), starting at y' and y'' respectively, will meet for the first time exactly at time t. Denote by D the probability that these random walks will ever meet:

$$D = \sum_{t=1}^{\infty} f_t. \tag{1.23}$$

Lemma 1.5. *Let x' and t' satisfy* (1.20). *Then*

$$\begin{aligned} \lim_{\varepsilon\to 0} \sum_{\substack{G\in\mathcal{G}_1([\tau/\varepsilon]+t',[r/\varepsilon]+x'):\\ G \text{ is a loop}}} J(G) &= \lim_{\varepsilon\to 0} \sum_{\substack{G\in\mathcal{G}_1([\tau/\varepsilon],[r/\varepsilon]):\\ G \text{ is a loop}}} J(G) \\ &= \tau e^{-\tau} D\rho_0(r+a\tau). \end{aligned} \tag{1.24}$$

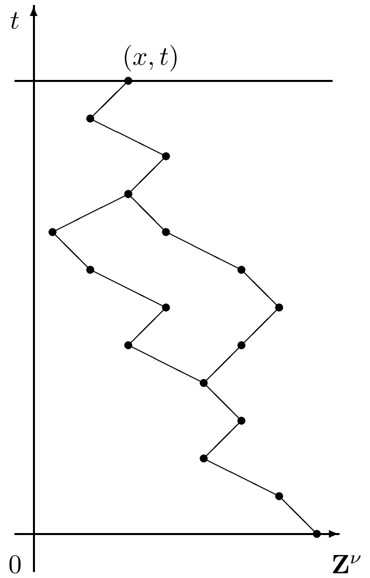

Figure 1

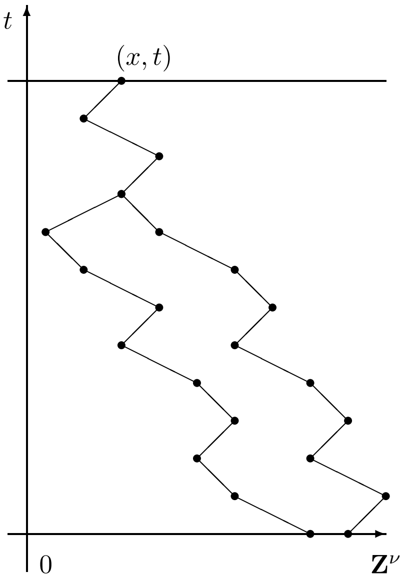

Figure 2

Proof. Rewrite the left-hand side of (1.24) in terms of random walks:

$$\sum_{\substack{G\in\mathcal{G}_1([\tau/\varepsilon],[r/\varepsilon]):\\ G \text{ is a loop}}} J(G) \;=\; \varepsilon(1-\varepsilon)^{[\tau/\varepsilon]} \sum_{t_1=1}^{[\tau/\varepsilon]} \sum_{x_1} P_{t_1}\big(\big[\frac{r}{\varepsilon}\big] \to x_1\big)$$

$$\cdot \sum_{t_2=1}^{[\tau/\varepsilon]-t_1-1} \sum_{x_2} f_{t_2}(x_2-x_1)\,(1-\varepsilon)^{t_2} \sum_{x_3} P_{[\tau/\varepsilon]-t_1-t_2}(x_2\to x_3)\rho_0(\varepsilon x_3). \quad (1.25)$$

Here $f_{t_2}(x_2)$ is the probability that the random walks $w^{(1)}$ and $w^{(2)}$, starting at y' and y'' respectively, meet for the first time at time t_2 at point x_2.

Decompose the summation over t_2 in (1.25) into two summations:

$$\sum_{t_2=1}^{[\tau/\varepsilon]-t_1-1} = \sum_{t_2=1}^{\phi(\varepsilon)} + \sum_{t_2=\phi(\varepsilon)+1}^{[\tau/\varepsilon]-t_1-1},$$

where $\phi(\varepsilon)$ is a function with the following properties:

$$\phi(\varepsilon)\to\infty, \quad \varepsilon\phi(\varepsilon)\to 0, \quad (\varepsilon\to 0). \quad (1.26)$$

From (1.25) it is easy to see that the sum $\sum_{t_2=\phi(\varepsilon)+1}^{[\tau/\varepsilon]-t_1-1}$ becomes small for ε sufficiently small. So, for proving the Lemma it is sufficient to consider the sum $\sum_{t_2=1}^{\phi(\varepsilon)}$ in (1.25).

Remark. The mean number of branchings before time $[\tau/\varepsilon]$ is finite. This means that branchings can not occur too fast: as a rule, the period of time between two successive branchings is of order ε^{-1}. Hence we can take $t_1 = [\tau_1/\varepsilon]$, for some τ_1, as the moment of the first branching.

More precisely, choose a small constant $\kappa > 0$ and decompose the summation over t_1 in (1.25) into two parts:

$$\sum_{t_1=1}^{[\tau/\varepsilon]} = \sum_{t_1=1}^{[\kappa/\varepsilon]} + \sum_{t_1=[\kappa/\varepsilon]+1}^{[\tau/\varepsilon]} . \tag{1.27}$$

The first sum does not exceed the probability that the branching process $\zeta_{\bar{t}}$ has a first branching before time $[\kappa/\varepsilon]$. This is less than $c\kappa$ for some constant c, depending only on τ.

Consider the second sum in the right-hand side of (1.27). Since t_1 is bigger than $[\kappa/\varepsilon]$, the particle moves according to the law of large numbers. This means that for each $\delta > 0$ and $\gamma > 0$ there exists $\varepsilon_0(\delta, \gamma)$, such that at time t_1 the moving particle is in a ball B_1 with centre $[r/\varepsilon] + \vec{a}t_1$ and radius δt_1, with a probability greater than $1 - \gamma$ for any $\varepsilon < \varepsilon_0$. During period t_2 (actually the loop) two moving particles can leave the ball B_1, but not further than $\phi(\varepsilon)\,\mathrm{diam}\,Q$. After this, during time $t_3 = [\tau/\varepsilon] - t_1 - t_2$, with probability close to 1 the particle moves to a point x_3 that belongs to the ball B_2 with centre $[r/\varepsilon] + \vec{a}t$ and radius $\delta(t_1 + t_3) + \phi(\varepsilon)\,\mathrm{diam}\,Q$. Hence, with probability arbitrarily close to 1 for sufficiently small ε we have

$$|\varepsilon x_3 - r - \vec{a}\tau| \le \delta\tau + \varepsilon\phi(\varepsilon)\,\mathrm{diam}\,Q, \tag{1.28}$$

for $\delta > 0$ arbitrary small, fixed. Choosing δ sufficiently small and using the continuity of ρ_0, we can replace $\rho_0(\varepsilon x_3)$ in the second sum in (1.27) by $\rho_0(r + \vec{a}\tau)$. Let us write

$$D_\varepsilon = \sum_{t_2=1}^{\phi(\varepsilon)} \sum_{x_2} f_{t_2}(x_2). \tag{1.29}$$

The second sum in (1.27) is equivalent to

$$\varepsilon \sum_{t_1=[\kappa/\varepsilon]+1}^{[\tau/\varepsilon]} D_\varepsilon (1-\varepsilon)^{[\tau/\varepsilon]} \rho_0(r + \vec{a}\tau) \;\longrightarrow\; (\tau - \kappa) D e^{-\tau} \rho_0(r + \vec{a}\tau). \tag{1.30}$$

Note that the left-hand side of (1.25) does not depend on κ. Thus, by taking the limit $\kappa \to 0$ and by using (1.30) we get the statement of Lemma 1.5. This argument does not change when we add x' and t' satisfying (1.20). □

Lemma 1.6. *Let x' and t' satisfy* (1.20)*. Then*

$$\begin{aligned}\lim_{\varepsilon\to 0} \sum_{\substack{G\in\mathcal{G}_1([\tau/\varepsilon]+t',[r/\varepsilon]+x'):\\ G \text{ is an actual branching}}} J(G) &= \lim_{\varepsilon\to 0} \sum_{\substack{G\in\mathcal{G}_1([\tau/\varepsilon],[r/\varepsilon]):\\ G \text{ is an actual branching}}} J(G)\\ &= (1-D)\rho_0^2(r+\vec{a}\tau)\int_0^\tau e^{-(\tau-\tau_1)}e^{-2\tau_1}d\tau_1.\end{aligned}$$

Proof. We have

$$\begin{aligned}&\sum_{\substack{G\in\mathcal{G}_1([\tau/\varepsilon],[r/\varepsilon]):\\ G \text{ is an actual branching}}} J(G)\\ &\quad= \varepsilon\,(1-\varepsilon)^{[\tau/\varepsilon]}\sum_{t_1=1}^{[\tau/\varepsilon]}(1-\varepsilon)^{[\tau/\varepsilon]-t_1}\sum_{x_1}P_{t_1}\big([\frac{r}{\varepsilon}]\to x_1\big)\\ &\qquad\cdot\sum_{x_2,x_3}\widetilde{P}_{[\tau/\varepsilon]-t_1-1}(x_1+y',x_1+y''\to x_2,x_3)\,\rho_0(\varepsilon x_2)\,\rho_0(\varepsilon x_3).\end{aligned}\tag{1.31}$$

Here $\widetilde{P}$ stands for the probability that two particles, starting at x_1+y' and x_1+y'' respectively, and performing two independent random walks with jump probabilities a_y, will be in x_2 and x_3 respectively at time $[\tau/\varepsilon]-t_1-1$, without their trajectories having intersected.

Using the same arguments as in Lemma 1.5 we can replace $\rho_0(\varepsilon x_2)\rho_0(\varepsilon x_3)$ in (1.31) by $\rho_0^2(r+\vec{a}\tau)$. Denote $u(\varepsilon)=[\tau/\varepsilon]-t_1-1$ and rewrite $\widetilde{P}$ in (1.31) as follows:

$$\begin{aligned}&\widetilde{P}_{u(\varepsilon)}(x_1+y',x_1+y''\to x_2,x_3)\\ &\quad= P_{u(\varepsilon)}(x_1+y'\to x_2)\,P_{u(\varepsilon)}(x_1+y''\to x_3)\\ &\qquad-\sum_{t'=1}^{u(\varepsilon)}\sum_{x_4}f_{t'}(x_4)\,P_{u(\varepsilon)-t'}(x_4\to x_2)\,P_{u(\varepsilon)-t'}(x_4\to x_3).\end{aligned}\tag{1.32}$$

In this expression t' is the time of the first intersection, x_4 the place of the first intersection and we have added and subtracted all pairs of intersecting trajectories. Using the arguments and results from the previous Lemmas we see that summing the right-hand side of (1.32) over x_2 and x_3 we get an expression equivalent to $1-D_\varepsilon$, as $\varepsilon\to 0$.

Returning to (1.31) we get

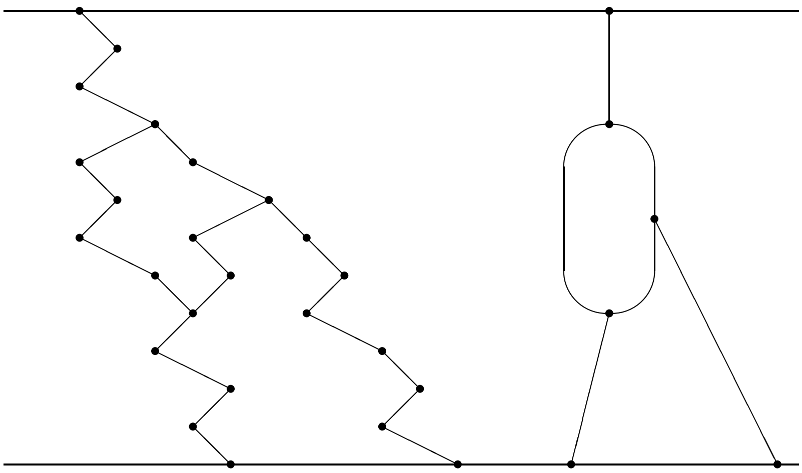

Figure 3

$$
\begin{aligned}
&\sum_{\substack{G\in\mathcal{G}_1([\tau/\varepsilon],[r/\varepsilon]):\\ G \text{ is an actual branching}}} J(G) \\
&\quad\sim\quad \varepsilon\,(1-\varepsilon)^{[\tau/\varepsilon]}\sum_{t_1=1}^{[\tau/\varepsilon]}(1-D_\varepsilon)\,(1-\varepsilon)^{[\tau/\varepsilon]-t_1}\,\rho_0^2(r+\vec{a}\tau) \\
&\quad\to\quad \rho_0^2(r+\vec{a}\tau)\,(1-D)\,e^{-\tau}\int\limits_0^\tau e^{-\tau_1}\,d\tau_1 \qquad (1.33)\\
&\quad=\quad (1-D)\,e^{-\tau}\,(1-e^{-\tau})\,\rho_0^2(r+\vec{a}\tau).
\end{aligned}
$$

This completes the proof of Lemma 1.6. □

3. Vanishing diagrams

Finally consider diagrams with more than one branching, $b \geq 2$. We say that a diagram has a *branching inside a loop*, if it contains a fragment as depicted in Figure 3, i.e. if one of the random walkers generating a loop, produces a daughter particle before coalescing.

Denote by $\widetilde{\mathcal{G}}_n$ the class of diagrams with n branchings but without a branching inside a loop. Then $\bigcup_{n\le N} (\mathcal{G}_n \setminus \widetilde{\mathcal{G}}_n)$ is the class of diagrams with $n \le N$ branchings and at least one branching inside a loop.

Lemma 1.7.

$$\lim_{\varepsilon\to 0} \sum_{G\in \bigcup_{n\le N} (\mathcal{G}_n\setminus\widetilde{\mathcal{G}}_n)} J(G) = 0. \tag{1.34}$$

Proof. Using the same argument as above, for (1.34) it suffices to consider only the summation over the diagrams having loops of size not larger than $\phi(\varepsilon)$. Denote this sum by $\sum' J(G)$ and let t be the first moment (in dual time) that a loop with a branching inside starts. Then

$$\sum{}' J(G) \le \sum_{t=1}^{[\tau/\varepsilon]} \mathsf{P}_\zeta(A_t), \tag{1.35}$$

where $\mathsf{P}_\zeta(A_t)$ is the probability that the branching process $\zeta_{\bar{t}}$ has at least two branchings: one exactly at time t, the other in the interval $(t, t+\phi(\varepsilon))$. Hence, the right-hand side of (1.35) does not exceed $\varepsilon^2([\tau/\varepsilon])\phi(\varepsilon)$ and by (1.26) this tends to 0 as $\varepsilon \to 0$. □

It follows from Lemma 1.7 that the only diagrams with a non-zero contribution in the limit, are trees with non-intersecting loops in the branches. We denote this class of the diagrams by $\widetilde{\mathcal{G}} = \bigcup_n \widetilde{\mathcal{G}}_n$. The graphs that correspond to these diagrams, have only trivial cycles i.e. two edges between the same pair of vertices (Figure 4). Indeed, if G has a branching inside a loop then Γ has a cycle at least of length 3 (see Figure 3) and vice versa.

Inductive decomposition of diagrams and the induction step

For each diagram $G([\tau/\varepsilon], x) \in \widetilde{\mathcal{G}}_n$ there are two possibilities:

A) There exist $x_1 \in \mathbf{Z}^\nu$, $1 \le t_1 \le [\tau/\varepsilon] - 2$, a diagram $G_1(t_1, x_1) \in \widetilde{\mathcal{G}}_{n-1}$ and a loop $G_2([\tau/\varepsilon]-t_1, x)$ with initial point x and final point x_1, such that $G([\tau/\varepsilon], x)$ is the union of the graphs $G_2([\tau/\varepsilon] - t_1, x)$ on $\{t_1, t_1 + 1, \dots, [\tau/\varepsilon]\} \times \mathbf{Z}^\nu$ and $G_1(t_1, x_1)$ on $\{0, 1, \dots, t_1\} \times \mathbf{Z}^\nu$.

Obviously, in this situation

$$J(G([\tau/\varepsilon], x)) = \widetilde{J}(G_2([\tau/\varepsilon] - t_1, x))J(G_1(t_1, x_1)) \tag{1.36}$$

with

$$\widetilde{J}(G) = J(G)/(\prod_y \rho_0(\varepsilon y)) :$$

the product is over all end-point vertices of G.

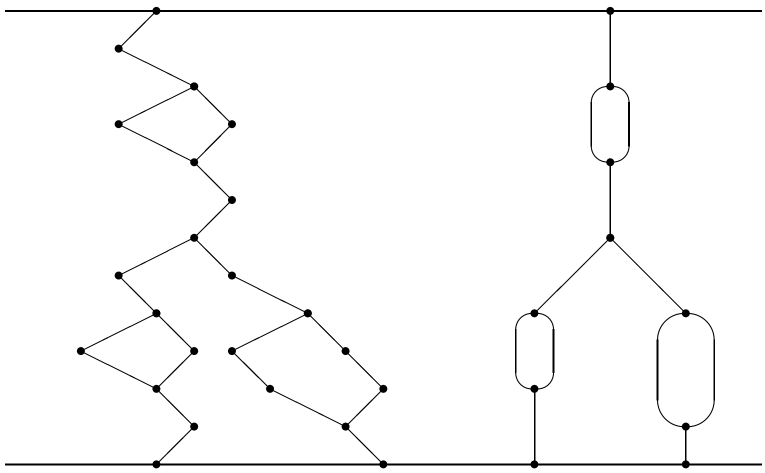

Figure 4

B) There exist
a point $(t_1, x_1) \in \mathbf{Z}_{[\tau/\varepsilon]-1} \equiv \{0, 1, \ldots, [\tau/\varepsilon]-1\} \times \mathbf{Z}^\nu$, $t_1 > 0$,
non-negative integers n_1, n_2, $n_1 + n_2 = n - 1$,
two diagrams $G_1(t_1 - 1, x_1 + y') \in \widetilde{\mathcal{G}}_{n_1}$ and $G_2(t_1 - 1, x_1 + y'') \in \widetilde{\mathcal{G}}_{n_2}$
and a path $G_0([\tau/\varepsilon] - t_1, x)$ from x to x_1,
such that diagram $G([\tau/\varepsilon], x)$ is obtained by putting G_0 on $\{t_1+1, \ldots, t\} \times \mathbf{Z}^\nu$, G_1 and G_2 both on $\mathbf{Z}_{t_1-1}$ and by subsequently adding two edges:

$$((t_1, x_1), (t_1 - 1, x_1 + y')) \text{ and } ((t_1, x_1), (t_1 - 1, x_1 + y'')).$$

Note that the diagrams G_1 and G_2 do not intersect, since $G([\tau/\varepsilon], x) \in \widetilde{\mathcal{G}}_n$.

In this case

$$J(G) = \varepsilon \widetilde{J}(G_0) J(G_1) J(G_2). \tag{1.37}$$

Using this decomposition we can construct any diagram inductively from the paths by choosing one of the already constructed diagrams and by subsequently adding a loop or by taking two of the already constructed diagrams and by subsequently connecting them through a new branching. The same is true for the corresponding graphs Γ (see Figures 5 and 6).

Consider an arbitrary graph Γ that corresponds to some diagram with n branchings and without any branching inside a loop. We want to show the

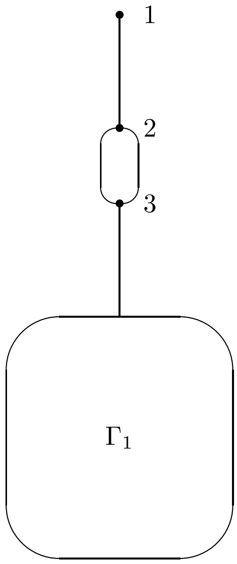

Figure 5

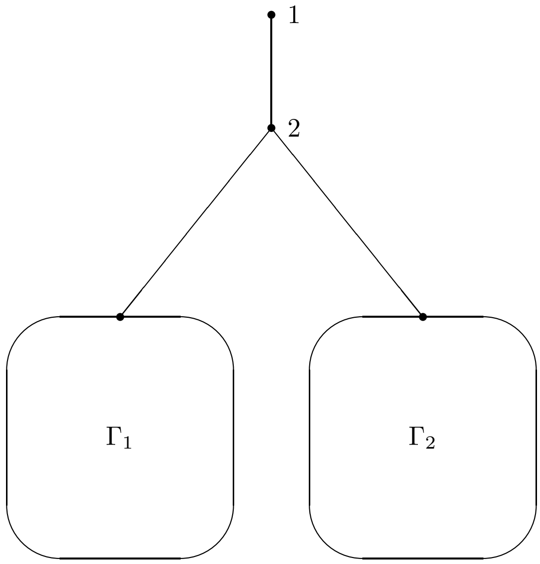

Figure 6

validity of the induction step, i.e. we want to prove that for each x' and t' satisfying (1.20) the following limit exists:

$$\lim_{\varepsilon\to 0} \sum_{G\,:\,\Gamma} J(G), \tag{1.38}$$

where the sum is over all diagrams G that correspond to the graph Γ with initial vertex $([\tau/\varepsilon]+t', [r/\varepsilon]+x')$, given that this limit exists for any Γ with $n-1$ branchings.

First consider case A) of the decomposition, i.e. there is a loop between vertices 2 and 3 of Γ (Figure 5).

Denote by Γ_1 the graph obtained from Γ by deleting vertices 1 and 2 and all incident edges. Then vertex 3 becomes the initial vertex of Γ_1. Each diagram G_1 corresponding to Γ_1 has $n-1$ branchings. Under the induction assumption for any such Γ_1 and for any x'_1 and t'_1 satisfying (1.20) the limit

$$\lim_{\varepsilon\to 0} \sum_{G_1\,:\,\Gamma_1} J(G) \;=\; L(\tau, r, \Gamma_1) \tag{1.39}$$

exists, where the sum is over all diagrams G_1 that correspond to Γ_1 with initial vertex $([\tau/\varepsilon]+t'_1, [r/\varepsilon]+x'_1)$.

Then

$$\sum_{G\,:\,\Gamma} J(G) \;\sim\; \varepsilon \sum_{t_1,z_1} (1-\varepsilon)^{t_1} P_{t_1}([r/\varepsilon]+x' \to z_1) \sum_{t_2=1}^{\phi(\varepsilon)} (1-\varepsilon)^{2t_2} \cdot \sum_{z_2} \widetilde{P}_{t_2}(z_1+y', z_1+y'' \to z_2, z_2) \sum_{G_1\,:\,\Gamma_1} J(G_1). \tag{1.40}$$

The rightmost sum in (1.40) is over all diagrams G_1 on $\mathbf{Z}_{[\tau/\varepsilon]-t_1-t_2}$ with initial vertex $([\tau/\varepsilon]-t_1-t_2, z_2)$ that correspond to the graph Γ_1. As in Lemma 1.5 we may take $t_1 = [\tau_1/\varepsilon]$ for some $0 < \tau_1 < \tau$. According to the law of large numbers we can write any z_2 in the sum $\sum_{z_2}$ as

$$z_2 = \big[\frac{r}{\varepsilon}\big] + \vec{a}\,\big[\frac{\tau_1}{\varepsilon}\big] + x'(\varepsilon),$$

for some $x'(\varepsilon)$ satisfying (1.20). This allows to apply the induction assumption to the rightmost sum in (1.40). The limit $L(\tau-\tau_1, r+\vec{a}\tau_1, \Gamma_1)$ of this sum exists and does not depend on z_2. Hence, using the definition of D for the sum $\sum_{t_2}\sum_{z_2}\widetilde{P}_{t_2}$ in (1.40), we see that the right-hand side of (1.40) tends to

$$D\int_0^{\tau} e^{-\tau_1} L(\tau-\tau_1, r+\vec{a}\tau_1, \Gamma_1)\, d\tau_1. \tag{1.41}$$

Next consider case B) of the decomposition (Figure 6). We have

$$\sum_{G\,:\,\Gamma} J(G) \;=\; \varepsilon \sum_{t_1,z} (1-\varepsilon)^{t_1} P_{t_1}\big(\big[\frac{r}{\varepsilon}\big]+x' \to z\big) \sum_{\substack{G_1\,:\,\Gamma_1,\; G_2\,:\,\Gamma_2,\\ G_1\cap G_2=\emptyset}} J(G_1)J(G_2). \tag{1.42}$$

Here the summation is over all diagrams G_1 on $\mathbf{Z}_{[\tau/\varepsilon]-t_1-1}$ with initial vertex $([\tau/\varepsilon]-t_1-1, z+y')$ that correspond to graph Γ_1 and over all diagrams G_2 on $\mathbf{Z}_{[\tau/\varepsilon]-t_1-1}$ with initial vertex $([\tau/\varepsilon]-t_1-1, z+y'')$ that correspond to graph Γ_2, such that G_1 and G_2 do not intersect. Let us add and substract in (1.42) the summation over all pairs G_1 and G_2 of intersecting diagrams that do not have any branching before their first intersection. By Lemma 1.7, the probability that G_1 and G_2 intersect after the first branching in G_1 or in G_2 tends to 0 as $\varepsilon \to 0$. Consequently,

$$\sum_{G\,:\,\Gamma} J(G) \sim$$

$$\sim\; \varepsilon \sum_{t_1,z} (1-\varepsilon)^{t_1} P_{t_1}\big(\big[\frac{r}{\varepsilon}\big]+x' \to z\big) \Bigg(\sum_{\substack{G_1\,=G_1(z+y',[\tau/\varepsilon]-t_1-1)\,:\,\Gamma_1,\\ G_2=G_2(z+y'',[\tau/\varepsilon]-t_1-1)\,:\,\Gamma_2}} J(G_1)\, J(G_2)$$

$$
- \sum_{t_2=1}^{\phi(\varepsilon)} \sum_{z_2} \widetilde{P}_{t_2}(z+y', z+y'' \to z_2, z_2) \sum_{\substack{G_1'(z_2,[\tau/\varepsilon]-t_1-1-t_2) \\ G_2'(z_2,[\tau/\varepsilon]-t_1-1-t_2)}} J(G_1')\, J(G_2') \Bigg). \tag{1.43}
$$

The summation over $G_1,\ G_2$ and $G_1',\ G_2'$ is free, i.e. there is no condition on intersection or non-intersection. As before, we can take $t_1 = [\tau_1/\varepsilon]$. According to the law of large numbers, for each $\delta,\ \gamma$ there exists ε_0, such that for any $\varepsilon < \varepsilon_0$

$$
\sum_{z \in B(\delta)} P_{t_1}\big(\big[\frac{r}{\varepsilon}\big] + x' \to z\big) > 1 - \gamma,
$$

where $B(\delta) \subset \mathbf{Z}^\nu$ is the ball with centre $[r/\varepsilon] + \vec{a}[\tau_1/\varepsilon]$ and radius $\delta[\tau_1/\varepsilon]$. Choosing δ small enough, we see that for any $z \in B(\delta)$ there exists $x' = x'(z)$, such that $\varepsilon x'(z) \to 0$ as $\varepsilon \to 0$ and

$$
z = \big[\frac{r}{\varepsilon}\big] + \vec{a}\,\big[\frac{\tau}{\varepsilon}\big] + x'(z).
$$

So we can apply the induction assumption to the sums over G_1 and G_2 in (1.43). Denote the limits of these sums by

$$
L(\tau - \tau_1, r + \vec{a}\tau_1, \Gamma_1) \quad \text{and} \quad L(\tau - \tau_1, r + \vec{a}\tau_1, \Gamma_2)
$$

respectively. By virtue of the induction assumption the limit for the sum over G_1' is equal to $L(\tau - \tau_1, r + \vec{a}\tau_1, \Gamma_1)$ and the limit of the sum over G_2' is equal to $L(\tau - \tau_1, r + \vec{a}\tau_1, \Gamma_2)$, because G_1' and G_1 correspond to the same Γ_1 and G_2' and G_2 correspond to the same Γ_2. Then

$$
\begin{aligned}
\sum_{G\,:\,\Gamma} J(G) \quad &\sim \quad \varepsilon(1-D) \sum_{t_1} (1-\varepsilon)^{t_1} \prod_{i=1}^{2} L(\tau - \tau_1, r + \vec{a}\tau_1, \Gamma_i) \\
&\longrightarrow \quad (1-D) \int_0^\tau e^{-\tau_1} \prod_{i=1}^{2} L(\tau - \tau_1, r + \vec{a}\tau_1, \Gamma_i)\, d\tau_1.
\end{aligned} \tag{1.44}
$$

This completes the proof of Lemma 1.3. □

Let us rearrange the series (1.19) by taking together the terms that correspond to the same graph Γ. As this series converges uniformly in ε, we can take the limits term by term. This yields

$$
\lim_{\varepsilon \to 0} \big\langle \xi_{[\tau/\varepsilon]}\big(\big[\frac{\tau}{\varepsilon}\big]\big)\big\rangle \;=\; e^{-\tau}\rho_0(r + a\tau) + \sum_{n=1}^{\infty} \sum_{\Gamma \in \mathcal{T}_n} L(\tau, r, \Gamma). \tag{1.45}
$$

This proves the first statement of Theorem 1.1 for $\alpha = 1,\ \beta = 0$. □

Let $\hat{\eta}_\tau$ be the following continuous time branching process living in $\mathbf{R}^\nu$. One particle starts from $r \in \mathbf{R}^\nu$ and moves straight with constant speed $\vec{a}$ during an exponential time with intensity 1. When the clock rings a coin is tossed: nothing happens with probability D, with probability $1-D$ the particle produces another particle that is placed at the same place as its mother. The particles are identical and they move and branch independently by the same laws, etc. Let $n(\hat{\eta}_\tau)$ be the total number of particles at time τ. It is clear from the proof of Lemma 1.3 that (1.45) is the expectation of the function $\rho_0(r+\vec{a}\tau)^{n(\hat{\eta}_\tau)}$ with respect to the distribution of the process $\hat{\eta}_\tau$.

Consider now the case $\beta \neq 0$. This case differs from the case $\beta = 0$ only in that some of the branches might not reach the 0-slice. The class $\mathcal{G}_0$ of diagrams without any branching consists now of two subclasses: the paths reaching the 0-slice and the paths dying before time $[\tau/\varepsilon]$. The limit of the second subclass is the sum of some geometric progression. Using this fact and Lemmas 1.4, 1.5 and 1.6 it is possible to explicitly calculate the limits of the diagrams in $\mathcal{G}_0$ and $\mathcal{G}_1$. Lemma 1.7 is true, since the proof does not use the value of the death probability for the dual process. The corresponding version of the inductive decomposition can be constructed as well, and so the main steps of the proof of Theorem 1.1 are the same.

1.5. The hydrodynamic equation

To complete the proof of Theorem 1.1 we need to show that the expression in (1.45) satisfies Equation (1.11). Note that the smoothness of $\rho(r,\tau)$ follows from the smoothness of $\rho_0(\cdot)$, the inductive decomposition of the diagrams and the explicit expressions for the limits of paths.

For the sake of brevity we describe the derivation for the case $\beta = 0$. The generalisation is quite evident. Let us denote the right-hand side of (1.45) by $F = F(\tau, r)$, and write

$$F_0(\tau, r) \; = \; e^{-\tau}\rho_0(r + a\tau).$$

Let us decompose the summation in (1.45) into two sums F_1 and F_2 as follows:

F_1 contains all Γ with the first event being the beginning of a loop,

F_2 contains all Γ with the first event being an actual branching.

Then by the inductive decomposition and Lemma 1.3

$$F_1(\tau, r) \quad = \quad \sum_{n=0}^{\infty} \sum_{\Gamma \in \mathcal{T}_n} D\rho_0^{n+1}(r + \vec{a}\tau) \int_0^\tau e^{-(\tau-\tau_1)} \widetilde{L}(\tau_1, r, \Gamma)\, d\tau_1,$$

$$F_2(\tau, r) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{\Gamma_1 \in \mathcal{T}_m} \sum_{\Gamma_2 \in \mathcal{T}_n} (1-D)\, \rho_0^{m+n+2}(r + \vec{a}\tau) \cdot \int_0^{\tau} e^{-(\tau-\tau_1)} \widetilde{L}(\tau_1, r, \Gamma_1) \widetilde{L}(\tau_1, r, \Gamma_2)\, d\tau_1,$$

where

$$\widetilde{L}(\tau_1, r, \Gamma) = L(\tau_1, r, \Gamma)/\rho_0^{n+1}(r + \vec{a}\tau).$$

Explicit calculation shows that

$$\frac{\partial F_0}{\partial \tau} = (\vec{a}, \nabla_r F_0) - F_0,$$

$$\frac{\partial F_1}{\partial \tau} = (\vec{a}, \nabla_r F_1) - F_1 + DF,$$

$$\frac{\partial F_2}{\partial \tau} = (\vec{a}, \nabla_r F_2) - F_2 + (1-D)F^2.$$

The hydrodynamic equation (1.12) immediately follows and so Theorem 1.1 is proved. □

2. Diffusion approximation

In this section we assume the random walk (1.3) to be such that the local central limit theorem holds. It is enough to assume for example, that the mentioned random walk is completely irreducible and aperiodic (see [5]).

Theorem 2.1. *Consider a nonlinear voter model* (1.1), (1.2) *with zero drift:*

$$\vec{a} = \sum_y y a_y = 0$$

and with initial product measure

$$\langle \xi_0(x) \rangle = \rho_0(\varepsilon^{1/2} x), \tag{2.1}$$

where ρ_0 is a smooth function. Let $B = (b_{ij})$ be the covariance matrix of the jumps of the random walk (1.3)*:*

$$b_{ij} = \sum_{y \in Q} a_y y^i y^j. \tag{2.2}$$

Then for any ν, for any $r \in \mathbf{R}^\nu$ and any $\tau \in \mathbf{R}_+$ the limit

$$\lim_{\varepsilon \to 0} \langle \xi_{[\tau/\varepsilon]}([r\varepsilon^{-1/2}]) \rangle = \rho(\tau, r) \tag{2.3}$$

exists and satisfies the equation

$$\frac{\partial \rho}{\partial \tau} = 1/2(B\nabla_r, \nabla_r)\rho - C\rho + \sum_{j=1}^{N} d_j \rho^j + \beta \tag{2.4}$$

with initial condition

$$\rho(0, r) = \rho_0(r). \tag{2.5}$$

The proof is analogous to the proof of Theorem 1.1. The dual process is the same. Since the mean number of branchings is bounded, it is possible to take the limit of the series with the graphical representation of the correlation function. The random walk (1.3) has zero mean, so we use the local central limit theorem instead of the law of large numbers. As above, the contributions of the diagrams having at least one branching inside a loop, vanish in the limit. The corresponding versions of Lemmas 4.8 and 4.11 are true. Note, that the limiting process $\hat{\eta}_\tau$ is the branching Brownian motion (possibly with death).

3. The law of large numbers

Let $\mathcal{S}(\mathbf{R}^\nu)$ denote the Schwartz space of rapidly decreasing smooth functions.

- For $\vec{a} \neq 0$ let $\xi_t(x)$ be the same voter model with a positive perturbation as in Section 1.1 with initial distribution (1.6). In this case we will consider the following generalised field:

$$S^\varepsilon_\tau(\varphi) = \varepsilon^\nu \sum_{x \in \mathbf{Z}^\nu} \xi_{[\tau/\varepsilon]}(x)\varphi(\varepsilon x), \quad \varphi \in \mathcal{S}(\mathbf{R}^\nu).$$

- In the case of "zero drift" $\vec{a} = 0$ we shall denote by $\xi_t(x)$ the process defined by (1.1) with initial product–measure (2.1) as in Section 2 and we introduce the generalised field:

$$T^\varepsilon_\tau(\varphi) = \varepsilon^\nu \sum_{x \in \mathbf{Z}^\nu} \xi_{[\tau/\varepsilon]}(x)\varphi(\varepsilon^{1/2} x), \quad \varphi \in \mathcal{S}(\mathbf{R}^\nu).$$

Theorem 3.1. *For any $\nu \geq 1$, any $\tau \geq 0$, and $\varphi \in \mathcal{S}(\mathbf{R}^\nu)$ we have the following convergence in mean for $\varepsilon \to 0$:*

$$S^\varepsilon_{[\tau/\varepsilon]}(\varphi) \longrightarrow \int \rho(\tau, r)\varphi(r)\, dr, \tag{3.1}$$

where $\rho(\cdot,\cdot)$ is the solution of hydrodynamical equation (1.8) *with initial condition* (1.9).

Theorem 3.2. *Let $\nu \geq 1$ and $\vec{a} = 0$. Then for any $\tau \geq 0$ and $\varphi \in \mathcal{S}(\mathbf{R}^\nu)$ we have the following convergence in mean for $\varepsilon \to 0$:*

$$T^\varepsilon_{[\tau/\varepsilon]}(\varphi) \longrightarrow \int \rho(\tau, r)\varphi(r)\, dr, \tag{3.2}$$

where $\rho(\cdot,\cdot)$ is the solution of hydrodynamical equation (2.4) *with initial condition* (2.5).

We will only give the proof of Theorem 3.1 because the proof of Theorem 3.2 is similar.

Proof of Theorem 3.1. The Theorem follows from two statements:

1. $\langle \xi_{[\tau/\varepsilon]}([r/\varepsilon])\rangle \overset{\varepsilon\to 0}{\longrightarrow} \rho(\tau, r)$ uniformly in r on compacta, for all τ.
2. $\mathrm{Var}(S^\varepsilon_\tau(\varphi)) \overset{\varepsilon\to 0}{\longrightarrow} 0$, for all $\varphi \in \mathcal{S}(\mathbf{R}^\nu)$.

Statement 1 is true by Theorem 1.1. To prove statement 2 we note that

$$\mathrm{Var}(S^\varepsilon_\tau(\varphi)) = \varepsilon^{2\nu} \sum_{z_1, z_2 \in \mathrm{Z}^\nu} \varphi(\varepsilon z_1)\overline{\varphi(\varepsilon z_2)} \cdot \mathrm{Cov}\big(\xi_{[\tau/\varepsilon]}(z_1), \xi_{[\tau/\varepsilon]}(z_2)\big).$$

Decompose the last sum into two parts:

$$\varepsilon^{2\nu} \sum_{z_1, z_2} = \varepsilon^{2\nu} \sum_{z_1} \sum_{z_2 : |z_2 - z_1| < \varepsilon^{-1/2-\delta}} + \varepsilon^{2\nu} \sum_{z_1} \sum_{z_2 : |z_2 - z_1| \geq \varepsilon^{-1/2-\delta}}, \tag{3.3}$$

where δ is a fixed constant with $0 < \delta < 1/2$.

The first sum is bounded by $O(\varepsilon^{(1/2-\delta)\nu})$ and hence tends to 0 as $\varepsilon \to 0$. Applying Lemma 3.1 below we find that the second sum in (3.3) vanishes as $\varepsilon \to 0$. This proves the Theorem. □

Lemma 3.1. *For any $\nu \geq 1$, $\tau \geq 0$, $0 < \delta < 1/2$*

$$\sup_{|z_2 - z_1| \geq \varepsilon^{-1/2-\delta}} \big|\mathrm{Cov}\big(\xi_{[\tau/\varepsilon]}(z_1), \xi_{[\tau/\varepsilon]}(z_2)\big)\big| \to 0, \quad \text{as } \varepsilon \to 0.$$

Proof. Let $\eta^{z_1,z_2}_\cdot$ be the dual process starting from the two-point configuration $\{z_1, z_2\}$. Consider the event

$$A^\varepsilon_\tau(z_1, z_2) = \left\{ \begin{array}{c} \text{two subtrajectories of the process } \eta^{z_1,z_2}_\cdot \\ \text{originating in } z_1 \text{ and } z_2 \text{ meet before time } \left[\frac{\tau}{\varepsilon}\right] \end{array} \right\}.$$

To prove the lemma it is sufficient to establish that

$$\mathsf{P}(A^\varepsilon_\tau(z_1, z_2)) \to 0 \text{ as } \varepsilon \to 0,$$

uniformly in $z_1, z_2 : |z_2 - z_1| \geq \varepsilon^{-1/2-\delta}$. To achieve this we note that for ε sufficiently small

1) the number of branchings in each subtrajectory is less than $\varepsilon^{-1/2}$ with probability at least $1 - c_1\varepsilon^{1/2}$ (by Chebyshev's inequality);

2) by the central limit theorem a walking particle $x(t) \in \mathbf{Z}^\nu$ with $x(0) = x_0$, $t \in [0, \varepsilon^{-1}\tau]$, does not deviate from the line $x_0 + at$ more than $\varepsilon^{-1/2-\delta/2}$ with probability at least $1 - c_2 \exp(-c_3\varepsilon^{-\delta/2})$;

3) the trajectories of branching random walk $\eta_\cdot^{x_0}$ starting at the single-point configuration $\{x_0\}$ do not deviate from the line $x_0 + at$ more than $\varepsilon^{-1/2-\delta/2}$ with probability at least

$$\left(1 - c_1\varepsilon^{1/2}\right)\left(1 - c_2 \exp(-c_3\varepsilon^{-\delta/2})\right)^{\varepsilon^{-1/2}},$$

which tends to 1 as $\varepsilon \to 0$.

Hence the subtrajectories under consideration do not meet with a probability that tends to 1, as $\varepsilon \to 0$, thus proving the Lemma. □

4. The voter model with general perturbations

4.1. The model and main results

The perturbed voter model is a discrete time conditionally independent Markov process ξ_t, $t \in \mathbf{Z}_+$, with state space $\mathcal{S} = \{0,1\}^{\mathbf{Z}^\nu}$ and conditional probabilities given by

$$\begin{aligned} \mathsf{P}\{\xi_{t+1}(x) = 1 | \xi_t(\cdot)\} \quad &= \quad (1-\varepsilon)\sum_{y\in Q} a_y \xi_t(x+y) \\ &\quad + \varepsilon\left(P_+(\xi_t(\cdot + x)) - P_-(\xi_t(\cdot + x)) + \beta\right), \end{aligned} \tag{4.1}$$

where $\varepsilon \geq 0$ is a small parameter, Q is a given finite subset of $\mathbf{Z}^\nu$, $a_y \geq 0$, $y \in Q$, $\sum_{y\in Q} a_y = 1$, and $P_+(\cdot), P_-(\cdot)$ are cylindrical functions on $\mathcal{S}$ given by

$$P_\pm(\xi(\cdot)) = \sum_{A\in\mathcal{P}_\pm, A=\{y_1,\dots,y_m\}} c_A^\pm \xi(y_1)\dots\xi(y_m),$$

where $c_A^\pm > 0$, $\mathcal{P}_+$ and $\mathcal{P}_-$ are given finite collections of finite subsets of the lattice $\mathbf{Z}^\nu$.

As in the positive perturbation case we assume that the random walk with jump probabilities (1.3) is *irreducible* and *aperiodic*.

Assumption on P_+ and P_-. In the sequel we assume that P_+ and P_- are such, that (4.1) varies between 0 and 1 for sufficiently small ε. Below we will consider only such ε. This is true, for example, if the following conditions are satisfied:

1. $\sum_{A\in\mathcal{P}_+} c_A^+ + \beta \le 1$,

2. for each $A \in \mathcal{P}_-$ there exists $y' \in A$ such that $y' \in Q$.

We need in the next subsections the following polynomials $f_+(u), f_-(u)$, $(u \in \mathbf{R})$:

$$f_\pm(u) = \sum_{j=1}^{N_\pm} d_j^\pm u^j, \tag{4.2}$$

where

$$\begin{aligned} N_\pm &= \max\{|A| \,:\, A \in \mathcal{P}_\pm\} \\ d_j^\pm &= \sum_{A\in\mathcal{P}_\pm, |A|\ge j} c_A^\pm \sum_{B_1\cup\ldots\cup B_j = A} D_{B_1,\ldots,B_j}. \end{aligned} \tag{4.3}$$

The inner sum in the last expression is taken over all different partitions of the set A into j non-empty non-intersecting subsets $B_1, \ldots, B_j$ $(B_i \cap B_k = \emptyset, i \ne k)$. $D_{B_1,\ldots,B_j}$ are defined in subsection 1.1.

Remark. We recall that in dimensions $\nu = 1, 2$ any symmetrical random walk with bounded jumps is *recurrent*. Hence the homogeneous random walk on $\mathbf{Z}^\nu$ with jump probabilities

$$q(x) = \sum_z p(z-x)p(z) \equiv \sum_{\substack{y_1\in Q, y_2\in Q:\\ y_1 - y_2 = x}} a_{y_1} a_{y_2} \tag{4.4}$$

is *irreducible, aperiodic* and *recurrent.* This implies that $D_A = 1$ for all $A \in \mathcal{P}_\pm$, and $D_{B_1,\ldots,B_j} \equiv 0$ for any $j > 1$ and all nonempty $B_1, \ldots, B_j$.

4.1.1. Euler limit

Let $\langle\cdot\rangle$ denote the expectation of the Markov process $\xi_t(\cdot)$ generated by (4.1) with initial distribution

$$\langle \xi_0(x_1)\ldots\xi_0(x_n)\rangle = \prod_{i=1}^n \rho_0(\varepsilon x_i), \tag{4.5}$$

where ρ_0 is a smooth function with $0 \le \rho_0(\cdot) \le 1$.

As in Section 1 we denote by $\vec{a}$ a mean jump vector for a discrete time random walk on $\mathbf{Z}^\nu$ with jump probabilities (1.3).

Theorem 4.1. *Let $\nu \ge 1$. There exists $\tau_0 > 0$ not depending on ν, such that for all $\tau \in [0, \tau_0)$ the limit*

$$\langle \xi_{[\tau/\varepsilon]}([r/\varepsilon])\rangle \longrightarrow \rho(\tau, r), \quad \varepsilon \to 0, \tag{4.6}$$

exists uniformly in r *on compacta.* $\rho(\tau, r)$ *is the unique solution of the equation*

$$\frac{\partial \rho}{\partial \tau} = (\vec{a}, \nabla_r \rho) - \rho + f_+(\rho) - f_-(\rho) + \beta \tag{4.7}$$

with initial condition $\rho(0, r) = \rho_0(r)$. *(The functions* $f_\pm(\cdot)$ *are defined in* (4.2) *and* (4.3).*)*

Remark. Note that under the conditions of this section the solution of Equation (4.7) exists for all $\tau > 0$ and it varies between 0 and 1, if the initial condition satisfies $0 \le \rho_0(\cdot) \le 1$. This follows by analysing the solutions of (4.7) using the method of characteristics.

Theorem 4.2. *Let* $\nu = 1, 2$. *Then for all* $\tau, 0 \le \tau < \infty$, *the convergence* (4.6) *is uniform on compacta and* $\rho(\tau, r)$ *is the unique solution of the following linear differential equation*

$$\frac{\partial \rho}{\partial \tau} = (\vec{a}, \nabla_r \rho) - \Big(1 - \sum_{A \in \mathcal{P}_+} c_A^+ + \sum_{A \in \mathcal{P}_-} c_A^-\Big)\rho + \beta \tag{4.8}$$

with initial condition $\rho(0, r) = \rho_0(r)$.

Remark. The assumptions on the aperiodicity of the random walk with jump probabilities (1.3) is convenient, but not essential. The method presented here allows us to treat the periodic case as well. In the latter case we can derive *non-linear* hydrodynamical equations in any dimension, including $\nu = 1, 2$.

Theorem 4.3. *Let* $\nu \ge 3$ *and let* $P_+(\cdot)$ *and* $P_-(\cdot)$ *be such, that the coefficients given in* (4.3) *satisfy the inequalities*

$$d_j^+ \ge d_j^- \quad \text{for all } j \ge 2. \tag{4.9}$$

Then the convergence result (4.6) *holds for all* τ, $0 \le \tau < \infty$, *uniformly on compacta.*

Remark. We hope that the conditions of Theorem 4.3 on $d_j^\pm$ are non-essential and that the statement of the Theorem is true without (4.9).

4.1.2. Diffusion limit

We consider the case of zero drift: $\vec{a} = 0$. Let $\langle \cdot \rangle$ denote the expectation operator of the process $\xi_t(\cdot)$ generated by (4.1) with initial distribution given by

$$\langle \xi_0(x_1) \dots \xi_0(x_n) \rangle = \prod_{i=1}^{n} \rho_0(\varepsilon^{1/2} x_i), \tag{4.10}$$

where $\rho_0 \in C^1(\mathbf{R}^\nu)$, $0 \le \rho_0(\cdot) \le 1$.

Consider the following second order parabolic differential equation:

$$\frac{\partial \rho}{\partial \tau} = \frac{1}{2} \sum_{i,j=1}^{\nu} b_{ij} \frac{\partial^2 \rho}{\partial r^i \partial r^j} - \rho + f_+(\rho) - f_-(\rho) + \beta \tag{4.11}$$

with initial condition

$$\rho(0, r) = \rho_0(r). \tag{4.12}$$

In (4.11) $B = (b_{ij})$ denotes the $\nu \times \nu$ – covariance matrix of the random walk with jump probabilities (1.3) (see also (2.2).) In this subsection we assume the local central limit theorem to hold [5] for this random walk.

Remark. If the assumptions on P_+ and P_- are satisfied then the solution of (4.11) exists for all $\tau > 0$ and $\rho_0 \in C^1(\mathbf{R}^\nu)$. Moreover, if $0 \le \rho_0(r) \le 1$ then $0 \le \rho(r, \tau) \le 1$ (see [9]).

The Theorems stated below are the corresponding versions of Theorems 4.1, 4.2 and 4.3 for the zero drift case. On the other hand they are the generalisations of the results of Section 2 for the case of non-positive perturbations.

Theorem 4.4. *Let* $\nu \ge 1$, $\vec{a} = 0$. *There exists* $\tau_0 > 0$ *independent of* ν *such that for all* $\tau \in [0, \tau_0)$ *the*

$$\langle \xi_{[\tau/\varepsilon]}([r/\varepsilon^{1/2}]) \rangle \longrightarrow \rho(\tau, r), \quad (\varepsilon \to 0), \tag{4.13}$$

limit exists uniformly in r *on compacta and* $\rho(\tau, r)$ *is the solution of equation* (4.11) *with initial condition* (4.12).

Theorem 4.5. *Let* $\nu = 1, 2$, $\vec{a} = 0$. *Then for all* τ, $0 \le \tau < \infty$, *the convergence result* (4.13) *holds uniformly on compacta and* $\rho(\tau, r)$ *is the unique solution of the linear second order differential equation*

$$\frac{\partial \rho}{\partial \tau} = \frac{1}{2} \sum_{i,j=1}^{\nu} b_{ij} \frac{\partial^2 \rho}{\partial r^i \partial r^j} - (1 - \sum_{A \in \mathcal{P}_+} c_A^+ + \sum_{A \in \mathcal{P}_-} c_A^-)\rho + \beta \tag{4.14}$$

with initial conditions (4.12).

Theorem 4.6. *Let* $\nu \ge 3$, $\vec{a} = 0$ *and* $P_+(\cdot)$ *and* $P_-(\cdot)$ *are such that coefficients given by* (4.3) *satisfy* (4.9).

Then the convergence result (4.13) *holds for all* $\tau, 0 \le \tau < \infty$, *uniformly on compacta.*

The proofs of Theorems 4.4, 4.5 and 4.6 do not need any additional technical tools in comparison with Theorems 4.1, 4.2 and 4.3. The only difference consists in the application of the local central limit theorem instead of the law of large numbers in order to explicitly evaluate the limiting hydrodynamical equation (4.11). We will give the proofs of Theorems 4.1, 4.2 and 4.3 in the next subsection.

4.1.3. Convergence of generalised fields

Theorem 4.7. *Let $\nu \geq 1$ and let τ_0 be the same as in Theorem 4.1. Then, if $\vec{a} \neq 0$, the convergence result (3.1) holds for all $\tau < \tau_0$ and all $\varphi \in \mathcal{S}(\mathbf{R}^\nu)$. If $\vec{a} = 0$ then the convergence result (3.2) holds for all $\tau < \tau_0$ and all $\varphi \in \mathcal{S}(\mathbf{R}^\nu)$.*

Theorem 4.8. *Suppose that $\nu = 1, 2$.*

If $\vec{a} \neq 0$, then for all $\tau \geq 0$ and all test functions $\varphi \in \mathcal{S}(\mathbf{R}^\nu)$ statement (3.1) is true with $\rho(\tau, r)$ the solution to (4.8) with initial condition $\rho(0, r) = \rho_0(r)$.

If $\vec{a} = 0$, then for all $\tau \geq 0$ and φ statement (3.2) is true with $\rho(\tau, r)$ the solution to equation (4.14) with initial condition (4.12).

Theorem 4.9. *Under the conditions of Theorems 4.3 and 4.6 the statement of Theorem 4.7 is valid for all $\tau \geq 0$.*

The proofs of the above three theorems are analogous to the proofs of Theorems 3.1 and 3.2. We only have to use Theorems 4.1–4.6 instead of Theorems 1.1 and 2.1 whenever this is needed. We omit the details.

4.2. Short times in any dimension

This subsection proves Theorem 4.1. The main difficulty of the non-positive perturbation case is the absence of a probabilistic dual process. We will show that the expectation of $\xi_{[\tau/\varepsilon]}([r/\varepsilon])$ can be represented by a series similar to a series in perturbation theory. In order to evaluate the limit of each term of this decomposition we first estimate the absolute values of each summand in the decomposition, whilst not taking into account possible cancellations between positive and negative summands. We show that the series for the first correlation function converges absolutely for sufficiently small τ.

4.2.1. Diagrams

For notational simplicity we will only consider the case

$$P_+(\xi(\cdot)) = \xi(y_1)\xi(y_2), \quad P_-(\xi(\cdot)) = \xi(y_3)\xi(y_4). \tag{4.15}$$

The generalisations are obvious.

Using (4.1) we can write the one-point correlation function at time t as a linear combination of one- and two-point correlation functions at time $t-1$.

$$\begin{aligned}\langle \xi_t(x) \rangle \quad = \quad & (1-\varepsilon) \sum_{y \in Q} a_y \langle \xi_{t-1}(x+y) \rangle \; + \; \varepsilon \, \langle \xi_{t-1}(x+y_1) \xi_{t-1}(x+y_2) \rangle \\ & - \varepsilon \, \langle \xi_{t-1}(x+y_3) \xi_{t-1}(x+y_4) \rangle.\end{aligned}$$

By using the conditional independence property we can rewrite the latter expression as a combination of correlation functions of order no more than 4 at time $t-2$. Note that $(\xi_s(z))^2 = \xi_s(z)$. Proceeding in such a way we finally obtain an expression for $\langle \xi_t(x) \rangle$ that only involves correlation functions at time 0. These are given by (4.5). Thus we obtain the decomposition of $\langle \xi_t(x) \rangle$ into the sum of contributions of *diagrams*:

$$\langle \xi_t(x) \rangle = \sum_{G \in \mathcal{D}(t,x)} J(G), \tag{4.16}$$

where the summation is taken over the set of diagrams with top vertex (t,x) (all definitions are given below).

Definition. A diagram G with top vertex (t,x) is a graph with vertices belonging to $\mathbf{Z}_+ \times \mathbf{Z}^\nu$ that is generated by the following *algorithm*. The algorithm has t steps.

1. First we construct the top vertex (t,x) on $\mathbf{Z}_+ \times \mathbf{Z}^\nu$.
2. As a second step we choose one of 3 possibilities:
   - We construct one of the vertices $(t-1, x+y), y \in Q$, and a line l connecting (t,x) and $(t-1,x+y)$. We assign to the line l weight $(1-\varepsilon)a_y$.
   - We construct a pair of vertices $(t-1, x+y_1)$ and $(t-1, x+y_2)$ and two lines connecting (t,x) with $(t-1,x+y_1)$ and $(t-1,x+y_2)$. We shall call this *pair* of lines a *fork*. We assign to this fork *weight* ε.
   - We construct a pair of vertices $(t-1, x+y_3)$ and $(t-1, x+y_4)$. After drawing two lines from (t,x) to $(t-1,x+y_3)$ and from (t,x) to $(t-1,x+y_4)$ we assign to this *fork* weight $(-\varepsilon)$.
3. Suppose that vertices $(t-s+1, x_1), \dots, (t-s+1, x_m)$ have been constructed in step s. In step $(s+1)$ we construct new vertices and lines or forks starting from *each* of the vertices $(t-s+1, x_j), j = 1, \dots, m$, using rule 2 of the algorithm. The assignments of the weights are the same as above. If some of the constructed vertices occupy the same points in $\mathbf{Z}_+ \times \mathbf{Z}^\nu$ we call them coalescing and we consider them as a single vertex in the next step.
4. The algorithm stops in step $t+1$. Let $(0, z_1), \dots, (0, z_k)$ be the vertices constructed in the last step. We call these vertices the *end-points* of a diagram and assign to them weights $\rho_0(\varepsilon z_1), \dots, \rho_0(\varepsilon z_k)$.

Definition. The contribution $J(G)$ of a diagram G that is generated by the above algorithm, is the product of the weights of the lines, forks and endpoints of G.

4.2.2. The majorising process

For notational simplicity we will only prove the Theorem for the case (4.15).

Let $\mathcal{D}^{(n)}(t,x)$ be the subset of all diagrams with n forks. Obviously,

$$\mathcal{D}(t,x) = \bigcup_{n=0}^{\infty} \mathcal{D}^{(n)}(t,x).$$

Next we state our main lemma. The proof will be given after the proof of Theorem 4.1.

Lemma 4.1. *There exists $\tau_0 > 0$ such that for all τ_1, $\tau_1 < \tau_0$, the following estimate holds*

$$\sup_{x\in \mathbf{Z}^\nu} \sup_{\tau\in[0,\tau_1]} \sum_{G\in\mathcal{D}^{(n)}([\tau/\varepsilon],x)} |J(G)| \le d^n \tag{4.17}$$

for some $d = d(\tau_1)$, $0 < d < 1$.

It is easy to complete the proof of Theorem 4.1. Using (4.16) and Lemma 4.1 we conclude that the series

$$\langle \xi_t(x)\rangle = \sum_{n=0}^{\infty} \sum_{G\in\mathcal{D}^{(n)}(t,x)} J(G) \tag{4.18}$$

converges uniformly in (ε, t, x) for $\varepsilon t \le \tau_1 < \tau_0$. Hence in order to evaluate the limit

$$\lim_{\varepsilon\to 0} \left\langle \xi_{[\tau/\varepsilon]}\left(\left[\frac{\tau}{\varepsilon}\right]\right)\right\rangle$$

it is sufficient to evaluate the limit of each term of the series (4.18). To do so we simply repeat the arguments used in the proof of the main theorem of Section 1. This proves Theorem 4.1. □

Proof of Lemma 4.1. We define the following discrete time random walk η_s, $s \in \mathbf{Z}_+$, on $\mathbf{Z}^\nu$, with branching and coalescing and starting at point $x \in \mathbf{Z}^\nu$. The state space of η_s is the set $\mathcal{M}$ of finite subsets of $\mathbf{Z}^\nu$, $\eta_0 = \{x\}$. Let $x_1, \dots, x_k$ be the state of the process at time s. Then at time $(s+1)$ the particles x_j, $j = 1, \dots, k$, evolve independently from each other by the following rules. Particle x_j

- jumps to site $(x_j + y), y \in Q$, with probability $\dfrac{1-\varepsilon}{1+\varepsilon}\, a_y$,
- produces two descendants at sites $(x_j + y_1)$ and $(x_j + y_2)$ (and dies itself) with probability $\dfrac{\varepsilon}{1+\varepsilon}$,

- produces two descendants at sites $(x_j + y_3)$ and $(x_j + y_4)$ (and dies itself) with probability $\dfrac{\varepsilon}{1+\varepsilon}$.

If some particles are at the same site at time $s+1$ they coalesce.

Let $F(\cdot) : \mathcal{M} \to [0,1]$ be the following function

$$F(M) = \prod_{z \in M} \rho_0(\varepsilon z),$$

where ρ_0 is the same as in (4.5). Consider the event

$$\{ \text{ the process } \eta_s \text{ has exactly } n \text{ branchings in the time interval } [0,t] \ \}.$$

Let $\chi_{n,t}(\eta.)$ be the indicator of this event. Then the following statement holds.

Lemma 4.2.

$$\sum_{G \in \mathcal{D}^{(n)}(t,x)} |J(G)| \le (1+\varepsilon)^{tn} \ \ \mathsf{E}_x \chi_{n,t}(\eta.) F(\eta_t),$$

where E_x denotes the expectation operator on the process $\eta.$ starting at $x \in \mathbf{Z}^\nu$.

Lemma 4.2 follows directly from the definition of the contributions of diagrams and the definition of the process η_s.

We associate with the process η_t a branching process $\zeta_t^{(\varepsilon)}$. It will be a Galton–Watson process, in which each particle produces one descendant with probability $(1-\varepsilon)/(1+\varepsilon)$ and two descendants with probability $2\varepsilon/(1+\varepsilon)$ per unit time. Hence for fixed $\varepsilon > 0$ the process $\zeta_t^{(\varepsilon)}$ exhibits a super-critical behaviour.

Since in the process $\zeta_t^{(\varepsilon)}$ particles can not coalesce, the following inequality holds

$$\mathsf{E}_x \chi_{n,t} \le \mathsf{P}\{\zeta_t^{(\varepsilon)} \ge n\}. \tag{4.19}$$

Lemma 4.1 follows from (4.19) and the next lemma.

Lemma 4.3. *For any $C > 0$ there exist $\tau_0 = \tau_0(C) > 0$ and $C_1 > 0$, such that for all $\tau \le \tau_0$ and for sufficiently small ε*

$$\mathsf{P}\{\zeta_{[\tau/\varepsilon]}^{(\varepsilon)} = n\} \le C_1 \cdot C^{-n}$$

uniformly in ε.

Proof of Lemma 4.3. Consider the generating function $\varphi_t^{(\varepsilon)}$ of the process $\zeta_t^{(\varepsilon)}$:

$$\varphi_t^{(\varepsilon)}(s) = (\underbrace{\varphi \circ \varphi \circ \ldots \circ \varphi}_{t})(s),$$

where

$$\varphi(s) = \frac{1-\varepsilon}{1+\varepsilon} s + \frac{2\varepsilon}{1+\varepsilon} s^2. \tag{4.20}$$

We shall use the following property of the functions $\varphi_t^{(\varepsilon)}(\cdot)$.

Lemma 4.4. *For any $s > 0$ let ε, t be such, that*

$$\frac{2\varepsilon}{1+\varepsilon} s < 1, \quad \varepsilon t < (2s)^{-1}.$$

Then

$$\varphi_t^{(\varepsilon)}(s) \leq \frac{1}{(2s)^{-1} - \varepsilon t}.$$

We will first complete the proof of Lemma 4.3.

It follows from Lemma 4.4 that for every $s > 1$ we can choose $\tau_0 = \tau_0(s) > 0$ such, that

$$\varphi_{[\tau/\varepsilon]}^{(\varepsilon)}(s) \leq C_1$$

for some constant $C_1 = C_1(\tau_0, s)$, uniformly in ε and $\tau \leq \tau_0$. Since

$$\varphi_{[\tau/\varepsilon]}^{(\varepsilon)}(s) = \sum_{n=1}^{\infty} \mathsf{P}\{\zeta_{[\tau/\varepsilon]}^{(\varepsilon)} = n\} s^n$$

the estimate

$$\mathsf{P}\{\zeta_{[\tau/\varepsilon]}^{(\varepsilon)} = n\} \leq \frac{C_1}{s^n}$$

immediately follows. This proves Lemma 4.3. □

Proof of Lemma 4.4. For notational simplicity we will omit the superscript (ε) in $\varphi_t^{(\varepsilon)}(\cdot)$. Remark that the functions $\varphi_t(\cdot), t \in \mathbf{N}$, constitute a recurrent sequence of the form

$$\varphi_t(s) - \varphi_{t-1}(s) = \frac{2\varepsilon}{1+\varepsilon} (\varphi_{t-1}^2(s) - \varphi_{t-1}(s))$$

for $t > 1$ and $\varphi_1(s) = \varphi(s)$ as in (4.20).

Let s be greater than 1. Consider the recurrent sequence $\psi_t(s)$, $t \in \mathbf{N}$, given by

$$\begin{aligned} \psi_t(s) - \psi_{t-1}(s) &= \frac{2\varepsilon}{1+\varepsilon} \psi_{t-1}^2(s), \quad t > 1, \\ \psi_1(s) &= \varphi_1(s). \end{aligned}$$

It is clear that $\varphi_t(s) \leq \psi_t(s)$ for all t. In turn, the sequence $\psi_t(s)$ is majorised by the solution of the differential equation

$$y' = \frac{2\varepsilon}{1+\varepsilon} y^2, \quad y(0) = \psi_1(s).$$

By solving the last equation we obtain the conclusion of Lemma 4.4. □

4.3. Convergence for all times: dimensions $\nu = 1, 2$

We will consider the case $\nu = 1, 2$ and we will give the proof of Theorem 4.2. We proceed as follows. The first step is a special renormalisation of the model. In the second step we consider the expansion of the one-point correlation function of the renormalised model into a series of contributions of diagrams, similar to the previous subsection. Next we show that the total contribution of all diagrams with at least one fork, tends to 0 as $\varepsilon \to 0$. So only diagrams without forks have a contribution in the limiting equation. The analysis is more delicate than the analysis in Subsection 4.2. This is why we are able to prove convergence to the hydrodynamical equation *for all* τ. We note that the linearity of Equation (4.8) reflects the fact that the symmetrical random walk in dimensions 1 and 2 is recurrent.

There are three possibilities for the perturbation terms $P_+(\cdot)$ and $P_-(\cdot)$ in (4.1):

A. $\sum_{A\in\mathcal{P}_+} c_A^+ > \sum_{A\in\mathcal{P}_-} c_A^-$;

B. $\sum_{A\in\mathcal{P}_+} c_A^+ = \sum_{A\in\mathcal{P}_-} c_A^-$;

C. $\sum_{A\in\mathcal{P}_+} c_A^+ < \sum_{A\in\mathcal{P}_-} c_A^-$.

We shall refer to them as cases A, B and C. First of all let us explain how to reduce cases A and C to case B. Denote

$$\alpha = \sum_{A\in\mathcal{P}_+} c_A^+ - \sum_{A\in\mathcal{P}_-} c_A^-.$$

The conditional probabilities (4.1) can be rewritten as

$$\begin{aligned} \mathsf{P}\{\xi_{t+1}(x) = 1 | \xi_t(\cdot)\} &= (1-\varepsilon') \sum_{y\in Q} a_y \xi_t(x+y) \\ &\quad + \varepsilon'(P'_+(\xi_t(\cdot + x)) - P'_-(\xi_t(\cdot + x)) + \beta'), \end{aligned} \tag{4.21}$$

where $\varepsilon' = (1-\alpha)\varepsilon$, $\beta' = \beta(1-\alpha)^{-1}$ and where in case A

$$\begin{aligned} P'_+(\xi(\cdot)) &= (1-\alpha)^{-1} P_+(\xi(\cdot)), \\ P'_-(\xi(\cdot)) &= (1-\alpha)^{-1}(P_-(\xi(\cdot)) + \alpha \sum_{y\in Q} a_y \xi(\cdot + y)), \end{aligned}$$

and in case C

$$P'_+(\xi(\cdot)) = (1-\alpha)^{-1}(P_+(\xi(\cdot)) - \alpha \sum_{y\in Q} a_y \xi(\cdot + y)),$$

$$P'_-(\xi(\cdot)) = (1-\alpha)^{-1} P_-(\xi(\cdot)).$$

The polynomials $P'_\pm(\cdot)$ defined above satisfy condition B. So the convergence in Theorem (4.2) for cases A and C follows from the convergence for case B. Simple calculations show that the limiting equations in cases A and C have the form (4.8) if the limiting equation in case B has this form.

Below we will only consider $P_\pm$ satisfying condition B. Consider the representation of $\langle \xi_{[\tau/\varepsilon]}([r/\varepsilon])\rangle$ as a sum of contributions of diagrams (see (4.16)):

$$\langle \xi_{[\tau/\varepsilon]}([\frac{\tau}{\varepsilon}])\rangle = \sum_{G\in\mathcal{D}([\tau/\varepsilon],[r/\varepsilon])} J(G) = \sum_{n=0}^{\infty} \sum_{G\in\mathcal{D}^{(n)}([\tau/\varepsilon],[r/\varepsilon])} J(G).$$

The Theorem follows immediately from the next two lemmas.

Lemma 4.5. *Let $P_\pm(\cdot)$ satisfy condition B. Then*

$$\sum_{n\geq 1} \sum_{G\in\mathcal{D}^{(n)}([\tau/\varepsilon],[r/\varepsilon])} J(G) \to 0, \quad (\varepsilon \to 0).$$

The convergence is uniform in τ on compacta in $\mathbf{R}_+$.

Lemma 4.6. *Under condition B the following convergence result holds (uni formly in τ on compacta):*

$$\sum_{G\in\mathcal{D}^{(0)}([\tau/\varepsilon],[r/\varepsilon])} J(G) \to \rho(\tau, r), \quad (\varepsilon \to 0).$$

$\rho(\tau, r)$ has the form

$$\rho(\tau, r) = e^{-\tau}\rho_0(r + \vec{a}\tau),$$

and is therefore the unique solution of the differential equation

$$\frac{\partial \rho}{\partial \tau} = (\vec{a}, \nabla_r \rho) - \rho$$

with initial condition $\rho(0, r) = \rho_0(r)$.

Before proving the main Lemmas 4.5 and 4.6 we introduce some notation and we state two technical lemmas that are of independent interest.

We denote by P^0 the probability law of one, two or more independent random walks with jump probabilities (1.3). Let further P be the probability law of a *coalescing* random walk. For example,

$$P((y_1, y_2) \xrightarrow{t} (z_1, z_2))$$

is the probability that two coalescing random walks starting in y_1 and y_2 do not meet before time t and they hit z_1 and z_2 respectively at time t. Analogously,

$$P((y_1, y_2) \xrightarrow{t} z)$$

is the probability that two walking particles starting in y_1 and y_2 coalesce into a single particle before time t and this particle hits z at time t.

Let us denote by $\sigma(y_1, y_2)$ the time for two walking particles with jump probabilities (1.3) and starting in y_1 and y_2 to coalesce. Then $\{\sigma(y_1, y_2) > t\}$ is the event

$$\left\{ \begin{array}{c} \text{two random walks } y_1(s) \text{ and } y_2(s), s \in \mathbf{Z}^1, \\ \text{starting in } y_1 \text{ and } y_2 \text{ respectively,} \\ \text{do not intersect for } 0 \leq s \leq t \end{array} \right\}.$$

Lemma 4.7. *Let* $\nu = 1, 2$. *Then*

(i)

$$P^0(\sigma(y_1, y_2) > t) \to 0, \quad t \to \infty;$$

(ii) *for any* $\tau > 0$

$$\varepsilon \sum_{t=1}^{[\tau/\varepsilon]} P^0(\sigma(y_1, y_2) \geq t) \to 0, \quad \varepsilon \to 0.$$

Remark. For $\nu = 1$ we have the estimate

$$P^0(\sigma(y_1, y_2) > t) \ \leq \ \frac{C}{(1+t)^{1/2}} \tag{4.22}$$

for some constant $C = C(y_1 - y_2) > 0$. See [18] for the proof. It is evident that (4.22) implies both statements of the above Lemma for $\nu = 1$.

Proof. The statement of Lemma 4.7(i) is in fact a restated recurrence property of the symmetrical random walk in dimensions 1 and 2. Statement (ii) of the Lemma follows from (i) and the following well-known theorem:

$$(\{a_n\},\ a_n \to 0,\ n \to \infty) \implies \Big(\frac{a_1 + \cdots + a_n}{n} \to 0,\ n \to \infty \Big).$$

□

Recall that in fact we consider a random walk with bounded jumps. This is because the set Q in (1.3) is finite. Let $d(Q)$ be the diameter of Q. The following lemma is true in any dimension.

Lemma 4.8. *For* $\nu \geq 1$, *any* $y_1, y_2 \in \mathbf{Z}^\nu$ *and* $t_1 \geq t_2 > 0$

$$\sum_z \big| P^0(y_1 \xrightarrow{t_1} z) - P^0(y_2 \xrightarrow{t_2} z) \big| \ \leq \ \frac{C \cdot (d(Q)(t_1 - t_2) + |y_1 - y_2|)}{(1+t_2)^{1/2}}$$

for some constant $C > 0$.

Lemma 4.8 can be proved by applying the local central limit theorem (cf. [5]).

Next we will prove Lemmas 4.5 and 4.6. First of all we note that the model under consideration has no dual probabilistic process as opposed to the non-perturbed voter model or the voter model with positive interaction. In other words, the contributions of the diagrams in the decomposition (4.16) can not be interpreted as the probabilities of the trajectories of some probabilistic process. So in order to prove the Theorem we have to take into account cancellations that might occur due to the presence of positive and negative forks in (4.16). On the other side, pieces of diagrams between neighbouring forks can be thought of as contributions of a "free dynamics", i.e. the process that is dual to the non-perturbed voter model. We note that the dual process of the non-perturbed voter model is the coalescing random walk with jump probabilities (1.3).

In order to simplify the notation and without loss of generality we will only prove the Theorem for the case

$$\begin{aligned} P_+(\xi(\cdot)) &= \xi(x_1)\,\xi(x_2), \\ P_-(\xi(\cdot)) &= \xi(x_3)\,\xi(x_4), \\ \beta &= 0. \end{aligned} \tag{4.23}$$

Proof of Lemma 4.5. First we consider

$$\begin{aligned} &\sum_{G\in\mathcal{D}^{(1)}([\tau/\varepsilon],[r/\varepsilon])} J(G) \\ &= \varepsilon\sum_{t_1=1}^{[\tau/\varepsilon]}\sum_w (1-\varepsilon)^{[\tau/\varepsilon]-t_1} P([r/\varepsilon] \xrightarrow{[\tau/\varepsilon]-t_1} w) \\ &\quad\cdot\Big[\sum_z U(t_1,w,z)\rho_0(\varepsilon z) + \sum_{z_1,z_2} R(t_1;w,z_1,z_2)\rho_0(\varepsilon z_1)\rho_0(\varepsilon z_2)\Big], \end{aligned} \tag{4.24}$$

where

$$\begin{aligned} R(t_1;w,z_1,z_2) &= (1-\varepsilon)^{2t_1}\Big(P((w+x_1,w+x_2)\xrightarrow{t_1}(z_1,z_2)) \\ &\quad - P((w+x_3,w+x_4)\xrightarrow{t_1}(z_1,z_2))\Big), \end{aligned}$$

and

$$U(t_1,w,z) = \sum_{s=0}^{t_1-2}(1-\varepsilon)^{2(t_1-s-1)+s}\sum_u \Pi(t_1,s,w,u)P(u\xrightarrow{s} z),$$

with

$$\begin{aligned} \Pi(t_1,s,w,u) &= P(\sigma(w+x_1,w+x_2)=t_1-s-1,(w+x_1)\xrightarrow{t_1-s-1}u) \\ &\quad - P(\sigma(w+x_3,w+x_4)=t_1-s-1,(w+x_3)\xrightarrow{t_1-s-1}u). \end{aligned}$$

The functions $U(\cdot\,,\cdot)$ in (4.24) correspond to the event that two walking particles that arose in a fork at time t_1, met before time 0 (in reverse time). The functions $R(\cdot\,,\cdot)$ in (4.24) correspond to the complementary event, i.e. that the two particles did not meet before 0.

From Lemma 4.7(ii) we obtain that the following expression tends to 0 as $\varepsilon \to 0$:

$$\varepsilon \sum_{t_1=1}^{[\tau/\varepsilon]} \sum_{z_1,z_2} |R(t_1; w, z_1, z_2)| \\ \leq \quad \varepsilon \sum_{t_1=1}^{[\tau/\varepsilon]} (P(\sigma(x_1,x_2) \geq t_1) + P(\sigma(x_3,x_4) \geq t_1)). \tag{4.25}$$

To estimate $U(\cdot,\cdot,\cdot)$ we first fix some positive integer valued function $\phi(\varepsilon)$, such that $\varepsilon\phi^2(\varepsilon) \to 0, \phi(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Note that

$$\varepsilon \sum_{t_1=1}^{\phi(\varepsilon)} \sup_w \sum_z |U(t_1, w, z)| \ \leq \ \varepsilon\phi(\varepsilon).$$

For $t_1 > \phi(\varepsilon)$ we decompose $U(\cdot,\cdot,\cdot)$ into two parts U_1 and U_2, the former of which corresponds to the event that two particles arising in a fork at time t_1 have coalesced (in reverse time) within the time interval $[t_1, t_1 - \phi(\varepsilon) - 1]$, and the latter of which corresponds to the opposite event, i.e.

$$U(t_1, w, z) = U_1(t_1, w, z) + U_2(t_1, w, z) = \sum_{s=t_1-\varphi(\varepsilon)-1}^{t_1-2} + \sum_{s=0}^{t_1-\varphi(\varepsilon)-2} .$$

Then $U_2(t_1, w, z)$ satisfies the inequality

$$\varepsilon \sum_{t_1=\phi(\varepsilon)+1}^{[\tau/\varepsilon]} \sup_w \sum_z |U_2(t_1, w, z)| \quad \leq \quad C\left(P(\sigma(x_1,x_2) \geq \phi(\varepsilon))\right. \\ \left. + \, P(\sigma(x_3,x_4) \geq \phi(\varepsilon))\right). \tag{4.26}$$

By virtue of Lemma 4.7 (i) the left-hand side of (4.26) tends to 0 as $\varepsilon \to 0$.

Let us consider $U_1(t_1, w, z)$. We can rewrite it as follows.

$$U_1(t_1, w, z) = W_1(t_1, w, z) + W_2(t_1, w, z),$$

where

$$W_i(t_1, w, z) = (1-\varepsilon)^{t_1} \sum_{s=t_1-\varphi(\varepsilon)-1}^{t_1-2} \alpha_i(\varepsilon, t_1 - s) \sum_u \Pi(t_1, s, w, u) P(u \xrightarrow{s} z),$$

and

$$\alpha_i(\varepsilon, t_1 - s) = \begin{cases} (1-\varepsilon)^{t_1-s-2} - 1 & \text{if } i = 1 \\ \\ 1 & \text{if } i = 2. \end{cases}$$

This yields the estimate for W_1:

$$\varepsilon \sum_{t_1=\phi(\varepsilon)+1}^{[\tau/\varepsilon]} \sup_w \sum_z |W_1(t_1, w, z)| \le 2\varepsilon \sum_{t_1=\phi(\varepsilon)+1}^{[\tau/\varepsilon]} \sum_{k=0}^{\phi(\varepsilon)+1} \alpha_1(\varepsilon, k) \ \le$$

$$\le 2\varepsilon \left[\frac{\tau}{\varepsilon}\right] \cdot \frac{(1-\varepsilon)^{\phi(\varepsilon)+2} - 1 + (\phi(\varepsilon)+2)\varepsilon}{\varepsilon} \ \le \ \varepsilon \left[\frac{\tau}{\varepsilon}\right] (\phi(\varepsilon)+2)^2 \varepsilon. \tag{4.27}$$

For estimating W_2 we use the following decomposition.

$$W_2(t_1, w, z) \ = \ W_{21}(t_1, w, z) + \ W_{22}(t_1, w, z)$$

where

$$W_{21}(\cdot) \quad = \quad (1-\varepsilon)^{t_1} \sum_{s=t_1-\varphi(\varepsilon)-1}^{t_1-2} \sum_u \Pi(t_1, s, w, u)$$

$$\cdot \Big(P(u \stackrel{s}{\to} z) \ - \ P((w + x_1) \stackrel{t_1-\phi(\varepsilon)-1}{\longrightarrow} z) \Big)$$

and

$$W_{22}(\cdot) = (1-\varepsilon)^{t_1} \left(\sum_{s=t_1-\varphi(\varepsilon)-1}^{t_1-2} \sum_u \Pi(t_1, s, w, u) \right) P((w + x_1) \stackrel{t_1-\phi(\varepsilon)-1}{\longrightarrow} z).$$

By Lemma 4.8 we have for some $C_1, C_2, C_3 > 0$

$$\begin{aligned} \varepsilon \sum_{t_1=\phi(\varepsilon)+1}^{[\tau/\varepsilon]} \sup_w \sum_z |W_{21}(t_1, w, z)| \quad &\le \quad 2\varepsilon \sum_{t_1=\phi(\varepsilon)+1}^{[\tau/\varepsilon]} \frac{C_1(d(Q)\phi(\varepsilon) + C_2)}{(1 + (t_1 - \phi(\varepsilon) - 1))^{1/2}} \\ &\le \quad C_3 \varepsilon \Big[\frac{\tau}{\varepsilon}\Big]^{1/2} \phi(\varepsilon). \end{aligned} \tag{4.28}$$

It is clear that

$$\begin{aligned} \sum_{s=t_1-\varphi(\varepsilon)-1}^{t_1-2} \sum_u \Pi(t_1, s, w, u) \ &= \ P(\sigma(x_1, x_2) \le \phi(\varepsilon)) - P(\sigma(x_3, x_4) \le \phi(\varepsilon)) \\ &= \ P(\sigma(x_3, x_4) > \phi(\varepsilon)) - P(\sigma(x_1, x_2) > \phi(\varepsilon)). \end{aligned}$$

In the last equality we have used the fact that the symmetrical one– and two–dimensional random walks with finite jumps are recurrent [18].

By Lemma 4.7(i) we get

$$\varepsilon \sum_{t_1=\phi(\varepsilon)+1}^{[\tau/\varepsilon]} \sup_w \sum_z |W_{22}(t_1,w,z)| \longrightarrow 0, \quad \varepsilon \to 0. \tag{4.29}$$

It follows from (4.25)–(4.29) that (4.24) tends to 0 as $\varepsilon \to 0$.

We have to prove the same statement for the sum

$$S_{\geq 2}\,(\tau, r\,;\varepsilon) \stackrel{\text{def}}{=} \sum_{m\geq 2} \sum_{G\in\mathcal{D}^{(m)}([\tau/\varepsilon],[r/\varepsilon])} J(G). \tag{4.30}$$

We can rewrite (4.30) as follows.

$$\begin{aligned} S_{\geq 2}\,(\cdot) \;&=\; \varepsilon^2 \sum_{t_1=2}^{[\tau/\varepsilon]} \sum_{t_2=1}^{[\tau/\varepsilon]-t_1} \sum_w (1-\varepsilon)^{[\tau/\varepsilon]-t_1} P([r/\varepsilon] \stackrel{[\tau/\varepsilon]-t_1}{\longrightarrow} w) \\ &\cdot \Bigg[\sum_z U(t_1-t_2,w,z)\big(\langle P_+(\xi_{t_2-1}(\cdot+z))\rangle - \langle P_-(\xi_{t_2-1}(\cdot+z))\rangle\big) \\ &+ \sum_{z_1,z_2} R(t_1-t_2;w,z_1,z_2)\Big((1-\varepsilon)\sum_{y\in Q} a_y \\ &\cdot \sum_{i=1}^{2} \Big\langle\, \xi_{t_2-1}(z_i+y)(P_+(\xi_{t_2-1}(\cdot+z_{3-i})) - P_-(\xi_{t_2-1}(\cdot+z_{3-i})))\,\Big\rangle \\ &+ \varepsilon \Big\langle\, \prod_{i=1}^{2}(P_+(\xi_{t_2-1}(\cdot+z_i)) - P_-(\xi_{t_2-1}(\cdot+z_i)))\,\Big\rangle\Big)\Bigg], \end{aligned} \tag{4.31}$$

with the same notation as in (4.23) and (4.24). In the above expression t_1 is the moment that the first fork arises and t_2 the moment that the second fork arises.

To estimate (4.31) we use the new summation index $v_1 = t_1 - t_2$ in (4.31) instead of t_1 and we change the order of summation:

$$\varepsilon^2 \sum_{t_1=2}^{[\tau/\varepsilon]} \sum_{t_2=1}^{[\tau/\varepsilon]-t_1} = \varepsilon \sum_{t_2=1}^{[\tau/\varepsilon]-1} \varepsilon \sum_{v_1=1}^{[\tau/\varepsilon]-t_2} .$$

The inner sum is analogous to the sum $\varepsilon\sum_{t_1}$ in (4.24) and can be estimated uniformly in t_2 by using (4.25)–(4.29). The outer sum $\varepsilon\sum_{t_2=1}^{[\tau/\varepsilon]-1}$ does not create any extra problem because of the factor ε in front:

$$\varepsilon \sum_{t_2=1}^{[\tau/\varepsilon]-1} 1 \leq \varepsilon\,\Big[\frac{\tau}{\varepsilon}\Big].$$

Hence the sum (4.30) tends to 0 as $\varepsilon \to 0$, thus completing the proof of Lemma 4.5. □

Proof of Lemma 4.6. None of the diagrams $G \in \mathcal{D}^{(0)}([\tau/\varepsilon],[r/\varepsilon])$ has a fork and hence it is the trajectory of a single particle walking back in time with jump probabilities (1.3) and starting in $[r/\varepsilon]$ at time $[\tau/\varepsilon]$. Consequently,

$$\sum_{G\in\mathcal{D}^{(0)}([\tau/\varepsilon],[r/\varepsilon])} J(G) = (1-\varepsilon)^{[\tau/\varepsilon]} \sum_z P^0\big([r/\varepsilon] \xrightarrow{[\tau/\varepsilon]} z\big)\,\rho_0(\varepsilon z).$$

The central limit theorem implies that for any $0<\alpha<1/2$

$$\sum_{z:|z-[r/\varepsilon]-\vec{a}[\tau/\varepsilon]|\le\varepsilon^{-1/2-\alpha}} P^0\big([r/\varepsilon] \xrightarrow{[\tau/\varepsilon]} z\big) \to 1, \quad \varepsilon\to 0.$$

The statement of the Lemma easily follows. □

4.4. High dimensions via auxiliary voter model

4.4.1. Auxiliary voter model

In the present section we prove Theorem 4.3. We consider the following class of perturbed voter models in dimensions $\nu\ge 3$.

$$\begin{aligned}\mathsf{P}\{\xi_{t+1}(x)=1|\xi_t(\cdot)\} &= (1-\varepsilon)\sum_{y\in Q} a_y\xi_t(x+y) \\ &\quad + \varepsilon\,(P_+(\xi_t(\cdot+x)) - P_-(\xi_t(\cdot+x))+\beta),\end{aligned} \tag{4.32}$$

with the cylindrical functions $P_\pm(\cdot)$ as described in Section 4.1. We assume that the following additional conditions are satisfied

$$d_j^+ \ge d_j^- \text{ for all } j\ge 2 \tag{4.33}$$

(see (4.3) for the notation).

Without loss of generality, to simplify notation and to avoid unessential details we will only consider the case

$$P_+(\xi(\cdot)) = \alpha_1\xi(x_1)\xi(x_2), \quad P_-(\xi(\cdot)) = \alpha_2\xi(x_3)\xi(x_4),$$

where $\alpha_1,\alpha_2>0$ and $x_1,x_2,x_3,x_4\in\mathbf{Z}^\nu$ are such that

$$\alpha_1 D_{\{x_1\},\{x_2\}} \ge \alpha_2 D_{\{x_3\},\{x_4\}}.$$

The latter condition is equivalent to (4.33).

Below we will use notation

$$D_{1,2}=D_{\{x_1\},\{x_2\}}, \quad D_{12}=D_{\{x_1,x_2\}}, \quad D_{3,4}=D_{\{x_3\},\{x_4\}}, \quad D_{34}=D_{\{x_3,x_4\}}.$$

Further define two constants $0 < \kappa \leq 1$ and $\mu \in \mathbf{R}$ as follows.

$$\kappa\alpha_1 D_{1,2} = \alpha_2 D_{3,4}, \quad \mu = \kappa\alpha_1 D_{12} - \alpha_2 D_{34}. \tag{4.34}$$

It is convenient for us to introduce an *auxiliary voter model*. It is a voter model with a positive perturbation. To define it we distinguish two cases.

1) Let $\mu \geq 0$. The auxiliary voter model is a discrete time conditionally independent Markov process with conditional probabilities

$$\mathsf{P}\{\xi_{t+1}(x) = 1 | \xi_t(\cdot)\} = (1-\varepsilon)\sum_{y\in Q} a_y \xi_t(x+y) + \varepsilon(P_{+,1}(\xi_t(\cdot + x)) + \beta), \tag{4.35}$$

where $P_{+,1}(\xi(\cdot)) = (1-\kappa)\alpha_1\xi(x_1)\xi(x_2) + \mu\xi(x_1)$. We will also use

$$\begin{aligned} P_{+,2}(\xi(\cdot)) &= \kappa\alpha_1\xi(x_1)\xi(x_2), \\ P'_{-}(\xi(\cdot)) &= \alpha_2\xi(x_3)\xi(x_4) + \mu\xi(x_1). \end{aligned}$$

Then the model (4.32) can be considered as a perturbation of the auxiliary model (4.35) with a perturbation term of the form

$$\begin{aligned} &\varepsilon\left(P_{+,2}(\xi_t(\cdot)) - P'_{-}(\xi_t(\cdot))\right) \qquad (4.36)\\ &\qquad = \varepsilon\left(\kappa\alpha_1\xi(x_1)\xi(x_2) - \alpha_2\xi(x_3)\xi(x_4) - \mu\xi(x_1)\right). \end{aligned}$$

2) Let $\mu < 0$. In this case the auxiliary voter model is defined by the conditional probabilities

$$\begin{aligned} \mathsf{P}\{\xi_{t+1}(x) = 1 | \xi_t(\cdot)\} &= (1-\varepsilon(1-\mu))\sum_{y\in Q} a_y \xi_t(x+y) \\ &\quad + \varepsilon(P_{+,1}(\xi_t(\cdot + x)) + \beta). \end{aligned} \tag{4.37}$$

We use the notation

$$\begin{aligned} P_{+,1}(\xi(\cdot)) &= (1-\kappa)\alpha_1\xi(x_1)\xi(x_2), \\ P_{+,2}(\xi(\cdot)) &= \kappa\alpha_1\xi(x_1)\xi(x_2) + |\mu|\sum_y a_y\xi(y). \end{aligned}$$

Hence in this case the model (4.32) can be considered as a perturbation of the auxiliary model (4.37) with perturbation term

$$\begin{aligned} &\varepsilon\left(P_{+,2}(\xi_t(\cdot)) - P_{-}(\xi_t(\cdot))\right) \\ &\qquad = \varepsilon\Big(\kappa\alpha_1\xi(x_1)\xi(x_2) + |\mu|\sum_y a_y\xi(y) - \alpha_2\xi(x_3)\xi(x_4)\Big). \end{aligned}$$

Since case 2 is by no means more difficult than case 1 we will *only* consider case 1 and the corresponding auxiliary voter model (4.35).

Let us note that the auxiliary voter model possesses a dual probabilistic process. Below we give a description of this duality.

4.4.2. Dual process of the auxiliary voter model

Let $\mathcal{F}(\mathbf{Z}^\nu)(\equiv \mathcal{F})$ be the set of all finite subsets of $\mathbf{Z}^\nu$ (including the empty set $\emptyset$). We define a discrete time homogeneous Markov process η_t with values in $\mathcal{F}(\mathbf{Z}^\nu)$ corresponding to a branching random walk with coalescing and death of particles. To be more concrete, let $\eta_t = B, B \in \mathcal{F}(\mathbf{Z}^\nu)$, be the state of the process at time t. Then at time $t+1$, each of the particles $x \in B$ behaves independently of the other particles by the following law:

- it jumps to site $x+y, y \in Q$, with probability $(1-\varepsilon)a_y$,
- it produces one descendant at site $(x+x_1)$ with probability $\kappa_0\varepsilon$ (and dies itself),
- it produces two descendants at sites $(x+x_1)$ and $(x+x_2)$ with probability $\alpha_0\varepsilon$ (and dies itself),
- it dies with probability $\varepsilon(1-\alpha_0-\kappa_0)$.

Two different particles coalesce, if the appear at the same moment in the same site in $\mathbf{Z}^\nu$; that is, they are considered to be a single particle from that time on.

The constants α_0 and κ_0 in the above description have the following values:

$$\alpha_0 = (1-\kappa)\alpha_1, \quad \kappa_0 = \mu.$$

It is easy to see that $\alpha_0, \kappa_0 \geq 0$, $\alpha_0 + \kappa_0 \leq 1$. If $\alpha_0 + \kappa_0 = 1$ then $\beta = 0$ and we take by definition $\dfrac{\beta}{1-\alpha_0-\kappa_0} = 0,\ 0^0 = 1.$

Duality relation. It can be easily established that the auxiliary voter model (4.35) and the process η_t are related by the identity

$$\left\langle \prod_{x\in A} \xi_t(x) \right\rangle_{\text{aux}} = \sum_{B\in\mathcal{F}(\mathrm{Z}^\nu)} \sum_{l\geq 0} P^{(\varepsilon)}(A \xrightarrow{t} B, N_d(t) = l)\, \rho_{0,\varepsilon}(B) \left(\frac{\beta}{1-\alpha_0-\kappa_0}\right)^l. \tag{4.38}$$

In this identity we use the following notation:

$P^{(\varepsilon)}$ is the probability law of η_t;

$A \xrightarrow{t} B$ denotes the event that the process $\eta_\cdot$ is in state B at time t given that at time 0 it is in state A;

$N_d(t)$ denotes the number of particles which have died in the time interval $\{1, 2, \ldots, t\}$,

$\rho_{0,\varepsilon}(B) = \prod_{z \in B} \rho_0(\varepsilon z)$.

4.4.3. General voter model as a perturbation of the auxiliary voter model

We need to represent the one–point correlation function $\langle \xi_t(x) \rangle$ for the model (4.32) as a power series in the perturbation term (4.36). The model (4.35) can be viewed as the "free dynamics" for such a decomposition. It is useful to think of such a decomposition as a resummation of a series like (4.16) for $\langle \xi_t(x) \rangle$. Below we give a constructive description of the mentioned decomposition.

The terms of the series can be obtained as the contributions of the diagrams that are constructed by the following algorithm. Each of the diagrams is a graph with a subset of $\mathbf{Z}_+ \times \mathbf{Z}^\nu$ as it's set of vertices and with top vertex (t, x).

1. We start with a particle in x at time t. This is the top vertex (t, x) of a diagram. In each of the subsequent steps of the algorithm we descend on a previous slice (for example from slice $\{s\} \times \mathbf{Z}^\nu$ to slice $\{s-1\} \times \mathbf{Z}^\nu$).

2. Let B be the configuration of particles at time $s, 0 < s \leq t$. $z \in B$ corresponds to vertex (s, z) of the diagram. Each particle $z \in B$ chooses one of the following possibilities:

   - to evolve (*backwards in time*) according the law of $\eta_\cdot$. In this case the corresponding part of the diagram is constructed as in Section 1.2.
   - to split at time $s-1$ into a pair of particles $(z + x_1)$ and $(z + x_2)$. In this case we construct two vertices $(s-1, z+x_1)$, $(s-1, z+x_2)$ and two lines connecting each of these with (s, z). We assign to this pair of lines weight $\varepsilon\alpha_1\kappa$ and we call it a *positive fork*.
   - to split at time $s-1$ into a pair of particles $(z+x_3)$ and $(z+x_4)$ with weight $(-\varepsilon\alpha_2)$ (so-called *negative fork*). In this case we construct new vertices and lines similar to the previous situation. We call this a *negative fork* with weight $(-\varepsilon\alpha_2)$.
   - to replace itself at time $s-1$ by a particle $(z + x_1)$ with weight $(-\varepsilon\kappa_0)$ (to have a unified terminology we call such a construction a *degenerate negative fork*). In this case we construct a vertex $(s - 1, z + x_1)$ and a line that connects it with (s, z).

3. The algorithm either stops on the 0–slice or if all particles died before time 0 by the law of $\eta_\cdot$.

In other words, a diagram is a graph consisting of parts of trajectories of the process $\eta_.$ connected by a number of forks.

The set of all diagrams with top vertex (t,x) is denoted by $\mathcal{H}(t,x)$ and the contribution $J(G)$ of the diagram G is defined as a product of probabilities of parts of the trajectories of the process $\eta_.$, of weights of forks and of multiplicative constants of the form

$$\left(\frac{\beta}{1-\alpha_0-\kappa_0}\right)^l \text{ and } \rho_{0,\varepsilon}(B),$$

where l is the number of particles that have died by the law of $\eta_.$ and B is the 0-slice configuration of particles (see also (4.38) for the notation). Note that forks only arise through the perturbation term. A branching that occurs in the evolution of the dual process of the auxiliary voter model *is not a fork.*

Let $\mathcal{H}^n(t,x)$ be the set of all diagrams with n forks (including degenerate ones). Then the sum

$$S_n(t,x) = \sum_{G\in\mathcal{H}^n(t,x)} J(G) \tag{4.39}$$

is the nth term of the decomposition of $\langle \xi_t(x)\rangle$ into powers of (4.36):

$$\langle \xi_t(x)\rangle = \sum_{n=0}^{\infty} S_n(t,x). \tag{4.40}$$

Remark. We note that for each finite t the series (4.40) is in fact a finite sum.

4.4.4. Main lemmas

Lemma 4.9. *For any $\tau>0, r\in\mathbf{R}^\nu$,*

$$\lim_{\varepsilon\to 0}\sum_{n\geq 1} S_n\big([\frac{\tau}{\varepsilon}],[\frac{r}{\varepsilon}]\big)=0$$

and the convergence is uniform in τ on compacta.

Lemma 4.10. *For any $\tau>0, r\in\mathbf{R}^\nu$, we have the following convergence uniformly in τ on compacta*

$$S_0\big([\frac{\tau}{\varepsilon}],[\frac{r}{\varepsilon}]\big)\to\rho(\tau,r), \quad \varepsilon\to 0,$$

where $\rho(\tau,r)$ is the unique solution to the differential equation

$$\begin{aligned}\frac{\partial\rho}{\partial\tau} \;=\; & (\vec{a},\nabla_r\rho) \;-\; (1-\alpha_1 D_{12}+\alpha_2 D_{34})\rho \\ & + (\alpha_1 D_{1,2}-\alpha_2 D_{3,4})\rho^2 \;+\; \beta\end{aligned}$$

with initial condition $\rho(0,r)=\rho_0(r)$.

It is clear from (4.40) that the statement of the Theorem immediately follows from the above Lemmas.

4.4.5. Important technical lemmas

For brevity let us introduce notation

$$P^{(\varepsilon)}(A \xrightarrow{s} B,\, l\,) = P^{(\varepsilon)}(A \xrightarrow{s} B,\, N_d(s) = l).$$

Lemma 4.11. *For $\nu \ge 1, v_1, v_2 \in \mathbf{Z}^\nu$ the following bound holds uniformly in $t, 0 \le t \le [\tau/\varepsilon]$*

$$\sum_{B\in\mathcal{F}(\mathrm{Z}^\nu)} \sum_{l\ge 1} |P^{(\varepsilon)}(v_1 \xrightarrow{t} B, l) - P^{(\varepsilon)}(v_2 \xrightarrow{t} B, l)| \;\le\; C|v_1 - v_2|\varepsilon^{1/2}, \quad (4.41)$$

where $C = C'\tau^{1/2}$ and C' is positive and depends only on the jump probabilities (1.3).

Proof. If $B = \emptyset$ then

$$P^{(\varepsilon)}(v_1 \xrightarrow{t} \emptyset) - P^{(\varepsilon)}(v_1 \xrightarrow{t} \emptyset) = 0$$

by the translation invariant property of $\eta_\cdot$.

If $B \ne \emptyset$ we introduce s to mark the time of the first branching of $\eta_\cdot$ with starting point one of the single point configurations v_1 or v_2. We have

$$\begin{aligned}
&P^{(\varepsilon)}(v_1 \xrightarrow{t} B, l) - P^{(\varepsilon)}(v_2 \xrightarrow{t} B, l)\\
&= \varepsilon \sum_{s=1}^{t} (1-\varepsilon)^{s-1} \sum_z (P^0(v_1 \xrightarrow{s-1} z) - P^0(v_2 \xrightarrow{s-1} z))\\
&\quad \cdot \Big[(1-\kappa)\alpha_1 P^{(\varepsilon)}((z+x_1, z+x_2) \xrightarrow{t-s} B, l) + \mu P^{(\varepsilon)}((z+x_1) \xrightarrow{t-s} B, l)\Big].
\end{aligned}$$

Substituting the last expression into (4.41) and applying Lemma 4.8 we get the following estimate

$$\varepsilon \sum_{s=1}^{t} \frac{C_1|v_1 - v_2|}{(1+s)^{1/2}} \;\le\; C|v_1 - v_2|\varepsilon t^{1/2},$$

for some constants $C_1, C > 0$. The statement of the Lemma easily follows. □

Let Z_t be a Galton–Watson branching process, in which each particle produces one or two descendants with probabilities $1-\varepsilon$ and ε respectively. We need the following property of this branching process, which can be proved by a direct calculation.

Lemma 4.12. *For each $\tau > 0$ there exists a constant $C(\tau)$, such that*

$$\sup_{0\le t\le[\tau/\varepsilon]} \mathsf{E} Z_t^3 \le C(\tau).$$

Let $|\eta_s|$ be the number of particles of the process $\eta_.$ at time s. Next Lemma is a simple consequence of Lemma 4.12.

Lemma 4.13. *For any $\delta > 0$ there exists a number $N = N(\tau, \delta) > 0$ independent of ε, such that*

$$\sup_{0 \le s \le [\tau/\varepsilon]} \mathsf{P}\{|\eta_s| > N\} < \delta.$$

Lemma 4.14. *Write $t_\varepsilon = [\tau/\varepsilon]$ and let $\nu \ge 1$.*

(i) *We have the following convergence result for $\varepsilon \to 0$*

$$\varepsilon \sum_{t=1}^{t_\varepsilon} \sum_{A \in \mathcal{F}(\mathrm{Z}^\nu)\setminus\{\emptyset\}} P^{(\varepsilon)}(0 \xrightarrow{t_\varepsilon - t} A) \sum_{B,l} \big| P^{(\varepsilon)}(A \xrightarrow{t} B, l) -$$

$$- \sum_{B_v} \sum_{l_v} \prod_{v \in A} P^{(\varepsilon)}(v \xrightarrow{t} B_v, l_v) \big| \longrightarrow 0. \tag{4.42}$$

The sum $\sum_{B_v}$ in (4.42) is taken over all $B_v \in \mathcal{F}$ such that $\bigcup_{v \in A} B_v = B$ and the sum $\sum_{l_v}$ is taken over all non-negative integers l_v with $\sum_{v \in A} l_v = l$.

(ii) *For any fixed $x_1, x_2 \in \mathbf{Z}^\nu$*

$$\varepsilon \sum_{t=1}^{t_\varepsilon} \sum_{A \in \mathcal{F}(\mathrm{Z}^\nu)\setminus\{\emptyset\}} P^{(\varepsilon)}(0 \xrightarrow{t_\varepsilon - t} A) \cdot$$

$$\cdot \sum_{w \in A} \sum_{B,l} \big| P^{(\varepsilon)}((A \setminus \{w\}) \cup \{w + x_1, w + x_2\} \xrightarrow{t} B, l) -$$

$$- \sum_{B', B_v} \sum_{l', l_v} \prod_{v \in A \setminus \{w\}} P^{(\varepsilon)}(v \xrightarrow{t} B_v, l_v) P^{(\varepsilon)}(\{w + x_1, w + x_2\} \xrightarrow{t} B', l') \big| \to 0 \tag{4.43}$$

as $\varepsilon \to 0$. The sum $\sum_{B', B_v}$ in (4.43) is taken over all elements of $\mathcal{F}$ with

$$B' \cup \bigcup_{v \in A \setminus \{w\}} B_v = B$$

and the sum $\sum_{l', l_v}$ is taken over all non-negative integers with

$$l' + \sum_{v \in A \setminus \{w\}} l_v = l.$$

Proof. Let us represent the probabilities in (4.42) as a sum of contributions of trajectories of the dual process. Taking into account possible cancellations in (4.42) we conclude that the left-hand side of (4.42) can be bounded by

$$\text{Const}\ \big(P^0\{\sigma(x_1,x_2)>\phi(\varepsilon)|\ \sigma(x_1,x_2)<\infty\}+\varepsilon\phi(\varepsilon)\big),$$

where $\phi(\varepsilon)$ is an integer valued function such that $\phi(\varepsilon)\to\infty$, $\varepsilon\phi(\varepsilon)\to 0$ as $\varepsilon\to 0$. This implies the statement (i) of Lemma 4.14. The proof of statement (ii) is similar. □

Lemma 4.15. *We have the following representation*

$$\sum_{u=1}^{\phi(\varepsilon)}(1-\varepsilon)^u P^0(\sigma(w_1,w_2)=u) \ = \ P^0(\,\sigma(w_1,w_2)\le\phi(\varepsilon)\,)+\beta(\varepsilon),$$

where $|\beta(\varepsilon)|\le C\varepsilon(\phi(\varepsilon))^2$.

Proof of Lemma 4.15. It is simple calculation (see (4.27)). □

4.4.6. Estimates for the non-surviving part: resummation of diagrams

We prove Lemma 4.9. First we prove that

$$S_1\big([\frac{\tau}{\varepsilon}],[\frac{r}{\varepsilon}]\big)\to 0,\quad \varepsilon\to 0. \tag{4.44}$$

The following notation is used.

$$t_\varepsilon=[\frac{\tau}{\varepsilon}],\ \ r_\varepsilon=[\frac{r}{\varepsilon}],\ \ \theta=\frac{\beta}{1-\alpha_0-\kappa_0}.$$

Taking into account (4.39) and the definition of diagrams we have

$$S_1(t_\varepsilon,r_\varepsilon)=\varepsilon\sum_{t_1=1}^{t_\varepsilon}\sum_{A\in\mathcal{F}\setminus\{\emptyset\}}\sum_{l}P^{(\varepsilon)}(r_\varepsilon\overset{t_\varepsilon-t_1}{\longrightarrow}A,\ l\,)\,\theta^l\sum_{B\in\mathcal{F}}U^\varepsilon(A,B,t_1)\,\rho_{0,\varepsilon}(B), \tag{4.45}$$

where we have denoted

$$U^\varepsilon(A,B,t_1) \ = \ \sum_{w\in A}\sum_{A_1\in\mathcal{F}}\sum_{l'}P^{(\varepsilon)}(A\setminus\{w\}\overset{1}{\longrightarrow}A_1,\ l'\,)\,\theta^{l'}$$

$$\cdot\sum_{l_1,l_2,l_3}\sum_{i=1}^{3}U_i^\varepsilon(w,A_1,B,l_i;t_1),$$

and

$$\begin{aligned} U_1^\varepsilon(w,A_1,B,l_1;t) = &\ \kappa\alpha_1 P^{(\varepsilon)}(A_1\cup\{w+x_1,w+x_2\} \stackrel{t}{\longrightarrow} B,\, l_1\,)\,\theta^{l_1}, \\ U_2^\varepsilon(w,A_1,B,l_2;t) = &\ -\alpha_2 P^{(\varepsilon)}(A_1\cup\{w+x_3,w+x_4\} \stackrel{t}{\longrightarrow} B,\, l_2\,)\,\theta^{l_2}, \\ U_3^\varepsilon(w,A_1,B,l_3;t) = &\ -\mu P^{(\varepsilon)}(A_1\cup\{w+x_1\} \stackrel{t}{\longrightarrow} B,\, l_3\,)\,\theta^{l_3}. \end{aligned} \tag{4.46}$$

Let $S_1'(t_\varepsilon,r_\varepsilon)$ be the same as the right-hand side of (4.45) except that the probabilities

$$(P^{(\varepsilon)}(A_1\cup\{w+x_{2i-1},w+x_{2i}\} \stackrel{t_1}{\longrightarrow} B,\, l_i\,),$$

(see (4.46)) are replaced by

$$\sum_{B_1\cup B_2=B}\ \sum_{l_{i1}+l_{i2}=l_i} P^{(\varepsilon)}(A_1 \stackrel{t_1}{\longrightarrow} B_1,l_{i1})\cdot P^{(\varepsilon)}(\{w+x_{2i-1},w+x_{2i}\} \stackrel{t_1}{\longrightarrow} B_2,l_{i2}),$$

for $i=1,2$ and for $i=3$ the probability $P^{(\varepsilon)}(A_1\cup\{w+x_1\} \stackrel{t_1}{\longrightarrow} B,\, l_3\,)$ is replaced by

$$\sum_{B_1\cup B_2=B}\ \sum_{l_{31}+l_{32}=l_3} P^{(\varepsilon)}(A_1 \stackrel{t_1}{\longrightarrow} B_1,l_{31})\cdot P^{(\varepsilon)}(\{w+x_1\} \stackrel{t_1}{\longrightarrow} B_2,l_{32}).$$

Otherwise speaking, consider the dual process starting at time t_1 from configuration $A_1\cup\{w+x_1,w+x_2\}$, $A_1\cup\{w+x_3,w+x_4\}$ or $A_1\cup\{w+x_1\}$ (see (4.46)). We let the particles of configuration $\{w+x_1,w+x_2\}$, $\{w+x_3,w+x_4\}$ or $\{w+x_1\}$ move independently of the particles of configuration A_1.

It follows from Lemma 4.14 that the limits

$$\lim_{\varepsilon\to 0} S_1(t_\varepsilon,r_\varepsilon) \text{ and } \lim_{\varepsilon\to 0} S_1'(t_\varepsilon,r_\varepsilon)$$

exist both or they do not and they are equal provided they exist. So, let us consider

$$\begin{aligned} S_1'(t_\varepsilon,r_\varepsilon) \quad = \quad & \varepsilon\sum_{t_1=1}^{t_\varepsilon}\ \sum_{A\in\mathcal{F}\setminus\{\emptyset\}}\ \sum_l P^{(\varepsilon)}(r_\varepsilon \stackrel{t_\varepsilon-t_1}{\longrightarrow} A,\, l\,)\,\theta^l \\ & \cdot\sum_{B_1,B_2}\ \sum_{w\in A}\ \sum_{l'} P^{(\varepsilon)}(A\setminus\{w\} \stackrel{t_1}{\longrightarrow} B_1,\, l'\,)\,\theta^{l'} \\ & \cdot\sum_{i=1}^{3}\sum_{l_i} T_i(w,B_2,l_i;t_1-1)\,\theta^{l_i}\,\rho_{0,\varepsilon}(B_1\cup B_2), \end{aligned} \tag{4.47}$$

where

$$\begin{aligned} T_1(w,B_2,l_1;t) &= \kappa\alpha_1 P^{(\varepsilon)}(\,(w+x_1,w+x_2) \stackrel{t}{\to} B_2,\, l_1\,), \\ T_2(w,B_2,l_2;t) &= -\alpha_2 P^{(\varepsilon)}(\,(w+x_3,w+x_4) \stackrel{t}{\to} B_2,\, l_2\,), \\ T_3(w,B_2,l_3;t) &= -\mu P^{(\varepsilon)}(\,(w+x_1) \stackrel{t}{\to} B_2,\, l_3\,). \end{aligned}$$

Further considerations are similar to the proof of Lemma 4.5. We need a better control of the dual process during the time interval $[t_1-1, t_1-\phi(\varepsilon)]$, where $\phi(\cdot)$ is a fixed integer valued function, such that

$$\phi(\varepsilon) \to \infty, \;\; \varepsilon^2\phi(\varepsilon) \to 0 \text{ as } \varepsilon \to 0.$$

Let us introduce the following events:

$H_i(w,t) =\{$ at least one of the particles of the process $\eta.$ that started at time t in configuration D_i has died or branched before time $t-\phi(\varepsilon)-1\}$, $i=1,2,3,$

with $D_i = \{w+x_{2i-1}, w+x_{2i}\}$ for $i=1,2$ and $D_3=\{w+x_1\}$;

$E_i(w,t) =\{$ the particles of the process $\eta.$ starting at time t in configuration D_i do not coalesce before time $t-\phi(\varepsilon)-1\} \cap \overline{H_i(w,t)}$, $i=1,2$;

$F_i(w,t) =\{$ the particles of the process $\eta.$ starting at time t in configuration D_i coalesce before time $t-\phi(\varepsilon)-1$ into a single particle$\}\cap\overline{H_i(w,t)}$, $i=1,2$.

We will write $T_{i,E},\ldots,T_{2,F},\ldots,T_{3,H},\ldots$ for quantities like the following:

$$T_{1,E}(w,z_1,z_2,l_1;s) = P^{(\varepsilon)}(\{(w+x_1,w+x_2) \stackrel{s}{\to} (z_1,z_2)\} \cap E_1(w,t),\, l_1\,).$$

Estimates for $T_{i,H}$.

Let $S'_{1,H}(t_\varepsilon, r_\varepsilon)$ be given by Equation (4.47) with T_i replaced by $T_{i,H}$. Since a particle in the dual process $\eta.$ branches or dies with probability of order ε at each moment of time, we have

$$\sum_{A\in\mathcal{F}\setminus\{\emptyset\}} P^{(\varepsilon)}(r_\varepsilon \stackrel{t_\varepsilon-t_1}{\to} A) \sum_{w\in A}\sum_{B\in\mathcal{F}}\sum_{i=1}^{3}\sum_{l_i} |T_{i,H}(w,B,l_i;t_1-1)|$$
$$\leq \text{ Const } \varepsilon\phi(\varepsilon), \qquad (4.48)$$

uniformly in $t_1, 0\le t_1\le t_\varepsilon$. Hence by (4.48) and Lemma 4.12 we conclude that $S'_{1,H}(t_\varepsilon, r_\varepsilon) \to 0$ as $\varepsilon\to 0$.

Estimates for $T_{3,\overline{H}}$.

For the sake of brevity we will use the notation: $t_\phi = t_1-\phi(\varepsilon)$. Let us represent $T_{3,\overline{H}}$ as

$$T_{3,\overline{H}}(\cdot) \;=\; T_{3,\overline{H},1}(\cdot) \;+\; T_{3,\overline{H},2}(\cdot)$$

where

$$T_{3,\overline{H},1}(w,B,l_3;t_1-1) = \;(-\mu)\sum_z P^{(\varepsilon)}((w+x_1) \stackrel{\phi(\varepsilon)-1}{\longrightarrow} z)$$
$$\cdot\Big(P^{(\varepsilon)}(z \stackrel{t_\phi}{\longrightarrow} B, l_3\,) - P^{(\varepsilon)}((w+x_1) \stackrel{t_\phi}{\longrightarrow} B, l_3\,)\Big),$$

$T_{3,\overline{H},2}(w,B,l_3;t_1-1) = \ (-\mu)P^{(\varepsilon)}((w+x_1) \stackrel{t_\phi}{\longrightarrow} B,\ l_3)).$

The following estimate holds

$$\varepsilon \sum_{t_1=1}^{t_\varepsilon} P^{(\varepsilon)}(r_\varepsilon \stackrel{t_\varepsilon - t_1}{\longrightarrow} A) \sum_{w\in A}\sum_{l_3}\sum_{B} |T_{3,\overline{H},1}(w,B,l_3;t_1-1)| \leq$$

$$\leq 2\varepsilon\mu \sum_{t_1=1}^{\phi(\varepsilon)} P^{(\varepsilon)}(r_\varepsilon \stackrel{t_\varepsilon - t_1}{\longrightarrow} A)|A| + \varepsilon\mu \sum_{t_1=\phi(\varepsilon)+1}^{t_\varepsilon} P^{(\varepsilon)}(r_\varepsilon \stackrel{t_\varepsilon - t_1}{\longrightarrow} A)|A| \cdot$$

$$\cdot \sup_{|z-w-x_1|\leq d(Q)\phi(\varepsilon)} \sum_{B}\sum_{l_3} |P^{(\varepsilon)}(z \stackrel{t_\phi}{\longrightarrow} B, l_3) - P^{(\varepsilon)}(\,(w+x_1) \stackrel{t_\phi}{\longrightarrow} B, l_3)|. \quad (4.49)$$

We have used the boundedness of jumps (1.3) in the above.

Applying Lemma 4.12 we conclude that the first summand in the right-hand side of (4.49) vanishes as $\varepsilon \to 0$. Using Lemmas 4.12 and 4.11 we conclude that the second summand in the right-hand side of (4.49) tends to 0 as $\varepsilon \to 0$.

Estimates for $T_{i,F}$.

The above arguments show that we really only need to consider the case that $t_1 > \phi(\varepsilon)$. In this case we can represent $T_{i,F}(\cdot)$ in the following way.

$$\begin{aligned} T_{1,F}(w,B,l_1;t_1-1) \ &= \ T_{1,F,1}(w,B,l_1;t_1-1) \ + \ T_{1,F,2}(w,B,l_1;t_1-1) \\ &\quad + T_{1,F,3}(w,B,l_1;t_1-1) \end{aligned} \quad (4.50)$$

where

$$\begin{aligned} T_{1,F,1}(\cdot) \ &= \ \kappa\alpha_1 \sum_{s=t_1-\phi(\varepsilon)}^{t_1-2} \sum_{v} (1-\varepsilon)^{2(t_1-s-1)} \\ &\quad \cdot P^0\left(\sigma(w+x_1,w+x_2) = t_1-s-1, (w+x_1) \stackrel{t_1-s-1}{\longrightarrow} v\right) \\ &\quad \cdot \sum_z (1-\varepsilon)^{s-t_\phi} P^0((v \stackrel{s-t_\phi}{\longrightarrow} z) \\ &\quad \cdot (P^{(\varepsilon)}(z \stackrel{t_\phi}{\longrightarrow} B, l_1) - P^{(\varepsilon)}((w+x_1) \stackrel{t_\phi}{\longrightarrow} B, l_1)), \\ T_{1,F,2}(\cdot) \ &= \ \kappa\alpha_1(1-\varepsilon)^{\phi(\varepsilon)+t_1-2} \\ &\quad \cdot \Bigg(\sum_{s=t_1-\phi(\varepsilon)}^{t_1-2} (1-\varepsilon)^{-s} P^0(\sigma(w+x_1,w+x_2) = t_1-s-1) \\ &\quad - P^0(\sigma(x_1,x_2) < \phi(\varepsilon)) \Bigg) P^{(\varepsilon)}((w+x_1) \stackrel{t_\phi}{\longrightarrow} B, l_1), \end{aligned}$$

$$T_{1,F,3}(\cdot) = \kappa\alpha_1(1-\varepsilon)^{\phi(\varepsilon)}P^0(\sigma(x_1,x_2)<\phi(\varepsilon)) \cdot P^{(\varepsilon)}((w+x_1)\stackrel{t_\phi}{\longrightarrow}B,l_1).$$

(For beavity we write $T_{1,F,j}(\cdot)$ instead of $T_{1,F,j}(w,B,l_1;t_1-1)$.)

Taking into account the boundedness of jumps in (1.3) and using Lemmas 4.11, 4.12 and 4.15 we note that

$$\varepsilon\sum_{t_1=1}^{t_\varepsilon}\sum_{A\in\mathcal{F}\setminus\{\emptyset\}}P^{(\varepsilon)}(r_\varepsilon\stackrel{t_\varepsilon-t_1}{\longrightarrow}A)\sum_{w\in A}\sum_{B}\sum_{l_1}\sum_{j=1}^{2}|T_{1,F,j}(w,B,l_1;t_1-1)| \qquad (4.51)$$

tends to 0 as $\varepsilon\to 0$.

The same considerations as in (4.50) and (4.51) are also valid for $T_{2,F}(\cdot)$. In the sequel we will use the quantity $T_{2,F,3}(w,B,l_1,t_1-1)\equiv T_{2,F,3}(\cdot)$:

$$T_{2,F,3}(\cdot) = -\alpha_2\,(1-\varepsilon)^{\phi(\varepsilon)}\,P^0\big(\sigma(x_3,x_4)<\phi(\varepsilon)\big)\,P^{(\varepsilon)}\big((w+x_1)\stackrel{t_\phi}{\longrightarrow}B,l_1\big).$$

Estimates for $T_{i,E}(\cdot)$.

We will consider the case $t_1>\phi(\varepsilon)$.

We will write $T_{1,E}(\cdot)$ instead of $T_{1,E}(w,B,l_1;t_1-1)$. Let us consider

$$\begin{aligned}T_{1,E}(\cdot) &= \kappa\alpha_1\,(1-\varepsilon)^{2(\phi(\varepsilon)-1)}\sum_{z_1,z_2}\\ &\cdot P^0\big((w+x_1,w+x_2)\stackrel{\phi(\varepsilon)}{\longrightarrow}(z_1,z_2),\sigma(w+x_1,w+x_2)\ge\phi(\varepsilon)\big)\\ &\cdot P^{(\varepsilon)}\big((z_1,z_2)\stackrel{t_\phi}{\longrightarrow}B,l_1\big).\end{aligned} \qquad (4.52)$$

As before we can replace the probabilities in the latter expression, namely

$$P^{(\varepsilon)}\big((z_1,z_2)\stackrel{t_\phi}{\longrightarrow}B,l_1\big)$$

by the sums

$$\sum_{B_1\cup B_2=B}\;\sum_{l_{11}+l_{12}=l_1}P^{(\varepsilon)}(z_1\stackrel{t_\phi}{\longrightarrow}B_1,l_{11})\cdot P^{(\varepsilon)}(z_2\stackrel{t_\phi}{\longrightarrow}B_2,l_{12}).$$

By virtue of Lemma 4.14 this substitution does not affect the limit of $S_1'(t_\varepsilon,r_\varepsilon)$.

After replacing these probabilities we can rewrite the expression thus obtained as

$$(1-\varepsilon)^{2(\phi(\varepsilon)-1)} \sum_{z_1,z_2} P^0\big((w+x_1, w+x_2) \xrightarrow{\phi(\varepsilon)} (z_1,z_2), \sigma(w+x_1,w+x_2) \geq \phi(\varepsilon)\big)\cdot$$

$$\cdot \sum_{B_1\cup B_2=B} \sum_{l_{11}+l_{12}=l_1} \Big[\big(P^{(\varepsilon)}(z_1 \xrightarrow{t_\phi} B_1, l_{11}) - P^{(\varepsilon)}((w+x_1) \xrightarrow{t_\phi} B_1, l_{11})\big)\cdot$$

$$\cdot P^{(\varepsilon)}(z_2 \xrightarrow{t_\phi} B_2, l_{12}) +$$

$$+ P^{(\varepsilon)}((w+x_1) \xrightarrow{t_\phi} B_1, l_{11})\big(P^{(\varepsilon)}(z_2 \xrightarrow{t_\phi} B_2, l_{12}) - P^{(\varepsilon)}((w+x_2) \xrightarrow{t_\phi} B_2, l_{12})\big)\Big] +$$

$$+ (1-\varepsilon)^{2(\phi(\varepsilon)-1)} P^0(\sigma(w+x_1,w+x_2) \geq \phi(\varepsilon))\cdot$$

$$\cdot \sum_{B_1\cup B_2=B} \sum_{l_{11}+l_{12}=l_1} P^{(\varepsilon)}((w+x_1) \xrightarrow{t_\phi} B_1, l_{11}) \cdot P^{(\varepsilon)}((w+x_2) \xrightarrow{t_\phi} B_2, l_{12}) \equiv$$

$$\equiv (\kappa\alpha_1)^{-1} T_{1,E,1}(\cdot) + (\kappa\alpha_1)^{-1} T_{1,E,2}(\cdot).$$

Taking into account the boundedness of jumps (1.3) and Lemmas 4.12 and 4.11 we obtain that the following expression tends to 0 as $\varepsilon \to 0$:

$$\varepsilon \sum_{t_1=1}^{t_\varepsilon} \sum_{A\in\mathcal{F}\setminus\{\emptyset\}} P^{(\varepsilon)}(r_\varepsilon \xrightarrow{t_\varepsilon - t_1} A) \sum_{w\in A} \sum_{B} \sum_{l_1} |T_{1,E,1}(w,B,l_1;t_1-1)|.$$

Analogous conclusions are valid for $T_{2,E}(\cdot)$.

We have proved that for evaluating the limit $\lim\limits_{\varepsilon\to 0} S_1(t_\varepsilon, r_\varepsilon)$ it is sufficient to evaluate the limit of

$$\varepsilon \sum_{t_1=\phi(\varepsilon)+1}^{t_\varepsilon} \sum_{A\in\mathcal{F}\setminus\{\emptyset\}} \sum_{l} P^{(\varepsilon)}(r_\varepsilon \xrightarrow{t_\varepsilon - t_1} A, l)\theta^l \sum_{w\in A} \sum_{B',l'} P^{(\varepsilon)}(A\setminus\{w\} \xrightarrow{t_1} B', l')\theta^{l'}\cdot$$

$$\cdot \Big[\sum_{B_1}\sum_{l_1} W_1(\varepsilon) P^{(\varepsilon)}((w+x_1) \xrightarrow{t_\phi} B_1, l_1)\theta^{l_1}\rho_{0,\varepsilon}(B'\cup B_1) +$$

$$+ \sum_{B_1,B_2} \sum_{l_1,l_2} W_2(\varepsilon) P^{(\varepsilon)}((w+x_1) \xrightarrow{t_\phi} B_1, l_1) P^{(\varepsilon)}((w+x_2) \xrightarrow{t_\phi} B_2, l_2)\cdot$$

$$\cdot \theta^{l_1+l_2}\rho_{0,\varepsilon}(B'\cup B_1\cup B_2)\Big],$$

where we have denoted

$$\begin{aligned} W_1(\varepsilon) &= \kappa\alpha_1(1-\varepsilon)^{\phi(\varepsilon)}P^0(\sigma(x_1,x_2)<\phi(\varepsilon)) \\ &\quad - \alpha_2(1-\varepsilon)^{\phi(\varepsilon)}P^0(\sigma(x_3,x_4)<\phi(\varepsilon)) - \mu, \\ W_2(\varepsilon) &= \kappa\alpha_1(1-\varepsilon)^{2\phi(\varepsilon)-1}P^0(\sigma(x_1,x_2)\geq\phi(\varepsilon)) \\ &\quad - \alpha_2(1-\varepsilon)^{2\phi(\varepsilon)-1}P^0(\sigma(x_3,x_4)\geq\phi(\varepsilon)). \end{aligned}$$

Noting that

$$P^0(\sigma(x_i,x_j)<\phi(\varepsilon)) \to D_{ij}, \quad P^0(\sigma(x_i,x_j)\geq\phi(\varepsilon)) \to D_{i,j}, \quad (1-\varepsilon)^{\phi(\varepsilon)} \to 1,$$

as $\varepsilon \to 0$, we get

$$\begin{aligned} \lim_{\varepsilon\to 0} W_1(\varepsilon) &= \kappa\alpha_1 D_{12} - \alpha_2 D_{34} - \mu \;=\; 0, \\ \lim_{\varepsilon\to 0} W_2(\varepsilon) &= \kappa\alpha_1 D_{1,2} - \alpha_2 D_{3,4} \;=\; 0. \end{aligned}$$

Consequently, $\lim_{\varepsilon\to 0} S_1(t_\varepsilon, r_\varepsilon) = 0$ and (4.44) is proved.

For proving the convergence

$$\sum_{n\geq 2} S_n(t_\varepsilon, r_\varepsilon) \longrightarrow 0, \quad \varepsilon \to 0$$

we should use the same ideas as in the proof of the corresponding part of Lemma 4.5. We therefore omit details. □

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